Lecture notes for Physics 10154: General Physics I

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Contents

List of Figures

Chapter 1

Introduction

Physics is a quantitative science that uses experimentation and measurement to advance our understanding of the world around us. Many people are afraid of physics because it relies heavily on mathematics, but don't let this deter you. Most physics concepts are expressed equally well in plain English and in equations. In fact, mathematics is simply an alternative short-hand language that allows us to easily describe and predict the behaviour of the natural world. Much of this course involves learning how to translate from English to equations and back again and to use those equations to develop new information.

1.1 The tools of physics

Before we begin learning physics, we need to familiarize ourselves with the tools and conventions used by physicists.

1.1.1 Scientific method

All sciences depend on the scientific method to advance knowledge in their fields. The scientific method begins with a hypothesis that attempts to explain some observed phenomenon. This hypothesis must be *falsifiable*, that is, there must exist some experiment which can disprove the hypothesis. The next step in the process is to design and perform an experiment to test the hypothesis. If the hypothesis does not correctly predict the results of the experiment, it is thrown out and a new hypothesis must be developed. If it correctly predicts the result of that particular experiment, the hypothesis is used again to make new predictions that can be experimentally tested. Although we will often speak of physical "laws" in this course, they are really all hypotheses that have been extensively compared to experiments and consistently correctly predict the result, but as our body of knowledge expands there is still the possibility that they may not be completely correct and so they will forever remain hypotheses.

1.1.2 Measurement

One of the fundamental building blocks of physics is measurement. Essentially, measurement assigns a numerical value to some aspect of an object. For example, if we want to compare the height of two people, we can have them stand side by side and we can easily see who is taller. What if those two people don't happen to be in the same place, but I still want to compare their heights? I can use some object, compare the height of the first person to that object, then compare the height of the second person to that object — this is measurement. This will only work, of course, if I use the same object to measure both people, i.e., I need to standardize the measurement in some way. I can do this by defining a fundamental *unit*.

Today, the entire world has agreed on a standard system of measurement called the SI (*système international*). Here are some of the fundamental units of the SI that you will encounter in this course:

Table 1.1: Prefixes used for powers of ten in the metric system

Power	**Prefix**	**Abbreviation**
10^{-18}	atto	a
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	μ
10^{-3}	milli	m
10^{-2}	centi	c
10^{-1}	deci	d
10^{1}	deka	da
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T
10^{15}	peta	P
10^{18}	exa	E

Fundamental unit for length is called the *meter* and is defined as the distance traveled by light in a vacuum during a time interval of 1/299 792 458 second.

Fundamental unit of mass is called the *kilogram* and is defined as the mass of a specific platinum-iridium alloy cylinder kept at the International Bureau of Weights and Measures in France.

Fundamental unit of time is called the *second* and is defined as 9 192 631 700 times the period of oscillation of radiation from the cesium atom.

The metric system builds on these fundamental units by attaching prefixes to the unit to denote powers of ten. Some of the prefixes and their abbreviations are shown in Table 1.1. According to this table then 1000 m (m is the abbreviation for meter) = 1 km = 100 dam = 100000 cm.

Scientific notation

Since it is cumbersome to read and write numbers with lots of digits, scientists use a shorthand notation for writing very small or very large numbers. Instead of writing 100000 cm as in the example above, I could write 1.0×10^5 cm where the $\times 10^5$ tells me to multiply by 100 000 (or move the decimal 5 places to the right). You might also sometimes see this written as 1.0e5 cm, which is the computer shorthand for $\times 10^5$.

Always remember to write down the units for any quantity. Without the units, we have no way to understand how you made the measurement!

1.1.3 Unit conversion

The fundamental units of the meter and kilogram are not the familiar units of feet or pounds that are typically used in the United States. You may occasionally be asked to convert from the imperial system (foot, pound) to the SI system (meters, kilograms). The method for unit conversion introduced here is also useful for converting between different units within a particular measurement system (i.e. meters to kilometers). If we know that

$$1 \text{ ft} = 0.3048 \text{ m},$$

we can rewrite this as

$$\frac{1 \text{ ft}}{0.3048 \text{ m}} = 1.$$

Now if we want to convert 5 m to feet, we can use the above ratio

$$5 \text{ m} \times \frac{1 \text{ ft}}{0.3048 \text{ m}} = 16.4 \text{ ft}$$

since we can multiply anything by 1 and it will remain the same. Note that I have set up the ratio so that the meters will cancel out. If I want to convert from feet to meters, I can simply invert the ratio (that's still 1) and use the same method.

Example: Converting speed

If a car is traveling at a speed of 28.0 m/s, is the driver exceeding the speed limit of 55 mi/h?

Solution: We will need to do two conversions here: first from meters to miles and then from seconds to hours. Using the method outlined above we can keep multiplying ratios until we get the units we want. First we need to know how many meters in a mile (there's a table in your textbook) 1 mi = 1609 m. Then we need the conversion factor for seconds to hours; this is usually done in two steps, 1 h = 60 min and 1 min = 60 s. Let's put it all together

$$28.0 \text{ m/s} \times \frac{1 \text{ mi}}{1609 \text{ m}} \times \frac{60 \text{ s}}{1 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ h}} = 62.2 \text{ mi/h}.$$

Yes, the driver is exceeding the speed limit.

Example: Converting powers of units

The traffic light turns green, and the driver of a high performance car slams the accelerator to the floor. The accelerometer registers 22.0 m/s^2. Convert this reading to km/min^2.

Solution: The same method will work here, but we just need to keep in mind that we will need to convert seconds to minutes twice because we have s^2. Remember that 1000 m=1 km and that 1 min = 60 s.

$$22.0 \text{ m/s}^2 \times \frac{1 \text{ km}}{1000 \text{ m}} \times \frac{60 \text{ s}}{1 \text{ min}} \times \frac{60 \text{ s}}{1 \text{ min}} = 79.2 \text{ km/min}^2.$$

The driver is accelerating at 79.2 km/min^2.

1.1.4 Dimensional analysis

Units can be handy when trying to analyze equations. Complicated formulas can be quickly checked for consistency simply by looking at the units (dimensions) of all the quantities to make sure both sides of the equation match. It is important to remember that the "=" symbol has a very specific meaning in mathematics and physics. It means that whatever is on either side of this sign **is exactly the same thing** even though it may look a little different on either side. If both sides must be the same, then they must also have the same units.

The basic strategy is to represent all quantities in the equation by their dimensions. For example, x is typically used to represent distance, so it will have dimension of length, $[x]$ = length = L, where the square brackets indicate that we are referring to the dimension of x. The variable t denotes time, so has dimension of time $[t]$ = time = T. Suppose we are given the equation

$$x = vt$$

where v is the speed of an object and has dimension of $[v] = L/T$. We can make sure that the equation is consistent by checking the dimensions. The left-hand side has dimension of length $[x] = L$. The right hand side is a little less obvious, but can be found with a little algebraic manipulation,

$$[vt] = \left(\frac{L}{T}\right)(T) = L.$$

So both sides of the equation have the same dimensions.

Example: Analysis of an equation

Show that the expression $v = v_0 + at$ is dimensionally correct where v and v_0 represent velocities, a is acceleration, and t is a time interval.

Solution: We know that the dimensions of velocity are $[v] = [v_0] = L/T$, so that part of the equation is fine. We just need to verify that the dimension of at has the same dimension as velocity.

$$[at] = \left(\frac{L}{T^2}\right)(T) = \frac{L}{T}.$$

The equation is dimensionally consistent.

1.1.5 Significant figures

When we make measurements, there are limits to the accuracy of the measurement. The accuracy of a measurement is denoted by the number of *significant figures* presented in a quantity. Significant figures are digits that are reliably known. Any non-zero digit is considered significant. Zeros are significant only when they are between two significant digits or they come after a decimal and a significant figure.

When doing calculations, your calculator will often give you a number with many digits as the result. **Most of these digits are not significant**. That is, the original numbers you were given do not allow you to be confident of the accuracy of most of those numbers, so **do not write them down in your answer.** There are rules for determining the number of significant figures you can keep after a calculation:

- When multiplying or dividing two or more quantities, the number of significant figures in the final product is the same as the number of significant figures in the *least* accurate of the factors being combined. For example, when multiplying 22.0 (three significant figures) and 2.0 (two significant figures), your answer will be 44 (two significant figures).
- When adding or subtracting two or more quantities, the number of decimal places in the result should equal the smallest number of decimal places of any term in the sum. For example, when adding 147 (zero decimal places) and 5.25 (two decimal places) the result is 152 (zero decimal places).

When dropping insignificant digits, remember to round up if the largest digit being dropped is 5 or greater and to round down if the largest digit being dropped is 4 or lower.

Example: Carpet calculation

Several carpet installers make measurements for carpet installation in the different rooms of a restaurant, reporting their measurement with inconsistent accuracy as shown in the table below. Compare the areas for the three rooms, taking into account significant figures. What is the total area of carpet required for these rooms?

	Length (m)	**Width (m)**
Banquet hall	14.71	7.46
Meeting room	4.822	5.1
Dining room	13.8	9

Solution: To find the area of each room, we multiply length by width. For the banquet hall, length has 4 significant figures and width has three, so our answer must also have three:

$$A_b = 14.71 \text{ m} \times 7.48 \text{ m} = 109.74 \text{ m}^2 \to 1.10 \times 10^2 \text{ m}^2.$$

For the meeting room, length has 4 significant figures and width has two, so our answer must also have two:

$$A_m = 4.822 \text{ m} \times 5.1 \text{ m} = 24.59 \text{ m}^2 \to 25 \text{ m}^2.$$

For the dining room, length has 3 significant figures and width has 1, so our answer must also have 1:

$$A_d = 13.8 \text{ m} \times 9 \text{ m} = 124.2 \text{ m}^2 \to 100 \text{ m}^2.$$

The total amount of carpet needed is the sum of all three areas.

$$A_t = A_b + A_m + A_d = 110 \text{ m}^2 + 25 \text{ m}^2 + 100 \text{ m}^2 = 235 \text{ m}^2.$$

Our least accurate area is the dining room with only 1 significant figure in the hundreds place, so the total area should be written as $A_t = 200 \text{ m}^2$.

1.1.6 Coordinate systems

Many aspects of physics deal with movement or locations in space. This requires the definition of a coordinate system. A coordinate system consists of:

- a fixed reference point called the origin, O;
- a set of *axes* or directions with a scale and labels;
- instructions on labeling a point in space relative to the origin and axes.

The number of axes in the coordinate system will depend on how many dimensions you need for your problem: a point on a line needs one coordinate, a point on a plane needs two coordinates, and a point in space needs three coordinates.

You may be familiar with the *Cartesian* or *rectangular* coordinate system. In two dimensions, we define x and y axes to emanate from the origin at right angles to each other (the x axis is usually horizontal and the y axis is usually vertical, see Fig. 1.4 in your textbook). A point, let's call it P, can be located with two numbers (x, y). Suppose P is at (3,5). This means that starting from the origin, we take 3 steps along the x-axis and 5 steps along the y-axis to get to P. In this system, positive numbers are to the right (for x) and up (for y) while negative numbers are to the left (for x) and down (for y).

We will sometimes also use the *polar* coordinate system. This system uses the coordinates (r, θ) defined as follows. Select the origin and a reference line (the reference line is usually what would be the x-axis in

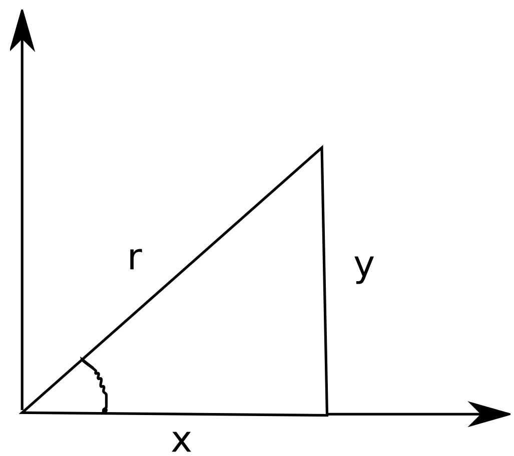


Figure 1.1: **Converting between Cartesian and polar coordinates.** Shown here are the x and y values for the Cartesian location of a point and the r and θ values for the polar location of that same point.

the Cartesian system, see Fig. 1.5 in your textbook). The point P is specified by the distance from the origin (r) and the angle θ between a line drawn from P to the origin and the reference line. In this system r will always be a positive number, while θ can be positive or negative. A positive value of θ is measured counterclockwise from the reference line and a negative value is measured clockwise from the reference line.

1.1.7 Trigonometry

In order to convert from the Cartesian to the polar coordinate system, you will need trigonometry. Trigonometry is a series of relationships between the sides and angles of a right angle triangle. The Cartesian x and y coordinates of a point form two sides of a right angle triangle whose hypotenuse is the polar coordinate r. The lower left angle of the triangle is the polar coordinate θ (see Fig. 1.1 or Fig. 1.6 in your textbook).

The basic trigonometric relationships are as follows:

$$\begin{aligned} \sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \\ \cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \\ \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}. \end{aligned} \tag{1.1}$$

There is also a relationship between the sides of the triangle,

$$r^2 = x^2 + y^2, \tag{1.2}$$

called the *Pythagorean theorem*.

Example: Converting coordinates

The Cartesian coordinates of a point in the xy-plane are $(x, y) = (-3.50\text{ m}, -2.50\text{ m})$. Find the polar coordinates of this point.

Solution: We are given x and y and we need to find r and θ. We have four possible equations to choose from (the trigonometric functions listed above). The Pythagorean theorem contains r, x, and y — we know two of those and want to find the other, so let's use that equation to find r.

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r &= \sqrt{x^2 + y^2} \\ r &= \sqrt{(-3.50\text{ m})^2 + (-2.50\text{ m})^2} \\ r &= 4.30\text{ m}. \end{aligned}$$

For θ, we have three possible equations. Since we now know x, y, and r, any of them will work. Let's use the equation for $\tan\theta$,

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ \tan\theta &= \frac{-3.50\text{ m}}{-2.50\text{ m}} \\ \tan\theta &= 0.714 \\ \theta &= \tan^{-1}(0.714) \\ \theta &= 35.5^\circ. \end{aligned}$$

We must be careful when using inverse trigonometric functions because there are two possible values of θ that give the same result for $\tan\theta$. In this case, your calculator guesses the wrong one (it assumes the first quadrant, while your point is in the third quadrant), so we must add 180° to get the correct angle, $\theta = 35.5^\circ + 180^\circ = 216^\circ$. The polar coordinates corresponding to the Cartesian point $(x, y) = (-3.50\text{ m}, -2.50\text{ m})$ are $(r, \theta) = (4.30\text{ m}, 216^\circ)$.

1.2 Problem solving

A large part of physics involves solving problems. There is a general strategy that can be used to tackle problems and it should be used throughout this course. The same basic step-by-step procedure can be used with some small variation for all problems.

1. Read the problem carefully, probably two or three times.
2. Draw a diagram to illustrate the quantities that are givn and those that you need to find.
3. Label the diagram with variables that represent the quantities you are given and those that you need to find. Make a list of the actual values of these quantities (the numbers you are given) beside the diagram. Set up a coordinate system and include it in the diagram.
4. Make a list of the variables you know and those that you don't know, but would like to find.
5. Make a list of the equations that might be useful. There need to be at least as many equations as unknowns.
6. Determine which equations you will actually use by comparing the equations to the known and unknown quantities.

7. Solve the equations for the unknown quantities. **Use algebra; do not plug in numbers until the very end**.
8. Substitute the known values to find a numerical result.
9. Check your answer to see if it makes sense. Are the units correct? Is the value reasonable?

Example: Round trip by air

An airplane travels $x = 4.50 \times 10^2$ km due east and then travels an unknown distance y due north. Finally, it returns to its starting point by traveling a distance of $r = 525$ km. How far did the airplane travel in the northerly direction?

Solution: Let's set the origin at the starting point of the airplane and set up a standard Cartesian coordinate system there. We are given that the plane travels $x = 4.50 \times 10^2$ km east, some unknown distance north and then returns to the starting point (we can assume a straight line here) traveling another $r = 525$ km. This circuit forms a right angle triangle where the lengths of two sides are given and the third side is the unknown. We know an equation that relates the three sides of a right angle triangle (Pythagorean theorem),

$$r^2 = x^2 + y^2.$$

We are given r and x, so we want to solve for y (using algebra, not numbers)

$$\begin{aligned} y^2 &= r^2 - x^2 \\ y &= \sqrt{r^2 - x^2}. \end{aligned}$$

Now, we can put in the numbers

$$\begin{aligned} y &= \sqrt{(525 \text{ km})^2 - (450 \text{ km})^2} \\ y &= 2.70 \times 10^2 \text{ km}. \end{aligned}$$

Chapter 2

Motion in one dimension

The universe is in constant motion — planets orbit stars, stars move in their galaxies, liquids, solids and gases move around on the surface of the planets and stars. It is no wonder then that one of the first concerns of physicists was the study of motion. The study of the *causes* of motion is called *dynamics*. The study of motion itself, without any concern for the cause, is called *kinematics*. We will begin our study of motion with the simplest possible case; the kinematics of motion in one dimension.

2.1 Displacement

Motion involves the displacement of an object from one location to another. An easy way to measure this is to set up an appropriate coordinate system. This coordinate system provides the *frame of reference* for the system under study. The position of an object is given by its coordinates in the frame of reference. As the object moves, its coordinates will change allowing us to track its motion.

The *displacement* of an object is defined as its change in position. Mathematically, this can be written as

$$\Delta x \equiv x_f - x_i, \tag{2.1}$$

where we use the symbol Δ (you will see this again) to mean *change in*, x_f denotes the final (f for final) position of the object, and x_i denotes the initial (i for initial) position of the object. It's important to remember that displacement is defined as the difference between final and initial positions, **it does not matter how you got there**.

From the definition of displacement, we can see that displacement can be positive (when x_f is bigger than x_i) or negative (when x_i is bigger than x_f). The sign gives us information about the *direction* of the car's motion. If we use a horizontal line with increasing values to the right to define our axis, then positive displacement is towards the right and negative displacement is towards the left. Because displacement has a magnitude (size) and a direction it is actually a *vector* quantity. This is in contrast to a *scalar* quantity which only has a magnitude. Vectors are usually denoted by boldface type and/or with an arrow over the variable. For example $\vec{x}$ denotes the magnitude and direction of displacement, whereas x simply denotes the magnitude of displacement.

2.2 Velocity

In everyday usage, velocity and speed are often used interchangeably. In physics, these are two different things — velocity is a vector quantity, it has both magnitude and direction, whereas speed is a scalar and only has a magnitude. If I say that I am driving a car at 60 mph, I am giving you the car's speed (magnitude only). If I say that I am driving a car at 60 mph to Dallas, I am giving you the car's velocity (magnitude and direction). There is also a more subtle distinction between the two. Speed is determined by the length of the path traveled by the object whereas velocity is determined by the displacement (which is independent of the path).

We define the *average speed* of an object as the length of the path it travels divided by the total elapsed time:

$$\begin{aligned} \text{Average speed} &\equiv \frac{\text{path length}}{\text{elapsed time}} \\ v &\equiv \frac{d}{\Delta t}. \end{aligned} \tag{2.2}$$

The *average velocity* of an object is defined as the displacement of the object divided by the total elapsed time:

$$\begin{aligned} \text{Average velocity} &\equiv \frac{\text{displacement}}{\text{elapsed time}} \\ \overline{v} &\equiv \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}. \end{aligned} \tag{2.3}$$

Note that average velocity can be either positive or negative (average speed is always positive) where, because we are in one dimension, the sign tells us the direction of travel.

Example: The confused football player

After receiving a kickoff at his own goal, a football player runs downfield to within inches of a touchdown, then reverses direction and races back until he's tackled at the exact location where he first caught the ball. During this run, which took 25 s, what is (a) the path length he travels, (b) his displacement, and (c) his average velocity in the x-direction? (d) What is his average speed?

Solution: Let's call the football player's starting position the origin. We are given that the football player runs 100 yd in the positive x direction and then runs 100 yd in the negative x direction. The round trip takes 25 s. We want several quantities: path length, displacement, average velocity, and average speed.
(a) The path length is the total distance actually travelled. He goes 100 yd in one direction and another 100 yd in the other direction, for a total of 200 yd.
(b) The displacement is determined only by the start and end positions of the football player. He starts at $x_i = 0$ (the origin) and ends at $x_f = 0$ (he returns to the same place). So the displacement is

$$\Delta x = x_f - x_i = 0.$$

(c) The average velocity is determined by the displacement; we can use the definition of average velocity given above.

$$\begin{aligned} \overline{v} &= \frac{\Delta x}{\Delta t} \\ \overline{v} &= \frac{0}{25\text{ s}} = 0. \end{aligned}$$

The average velocity is 0 yd/s. (d) The average speed is determined by the path length, which we found to be 200 yd.

$$\begin{aligned} v &= \frac{d}{\Delta t} \\ v &= \frac{200\text{ yd}}{25\text{ s}} = 8\text{ yd/s}. \end{aligned}$$

2.2.1 Graphical interpretation of velocity

We can plot the position of a moving object as a function of time. We put time on the horizontal axis (x-axis) and position on the vertical axis (y-axis) to get a *position vs. time* graph. If the car is moving at a constant velocity, the resulting curve will be a straight line with the average velocity given by the slope of that line (see Fig. 2.5 in your textbook). In general, though, most objects do not always move at a constant velocity and the position vs. time curve will be more complicated. We can still use the curve to find the average velocity of the object. The average velocity of an object during the time interval Δt is equal to the slope of the straight line joining the initial and final points on a graph of the object's position versus time.

2.2.2 Instantaneous velocity

So far, we have discussed average velocities which are measured over some, usually long, period of time. What is usually of more interest is the *instantaneous* velocity of an object, i.e. how fast was your car going the second the police pointed the radar gun at it. The instantaneous velocity is calculated using the concept of a *limit*. The formal definition is

$$v \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}. \tag{2.4}$$

If you recall the graphical representation of average velocity, we can find the average velocity by drawing a line between the two endpoints and finding the slope of that line. The idea of a limit simply means that we bring those two endpoints closer and closer together until they are essentially the same point. If the two endpoints are initially far apart, the slope of the line connecting them will probably change a lot as we move them closer together, but when the two points are close enough, the slope of the line will converge to some fixed value — this is the instantaneous velocity. Mathematically, the slope of the line tangent to the position versus time curve at a particular time is defined to be the instantaneous velocity at that time (with direction given by direction of travel). The instantaneous speed is the magnitude of the instantaneous velocity.

Example: Slowly moving train

A train moves slowly along a straight portion of track according to the graph of position vs. time in Fig. 2.7a of your textbook. Find (a) the average velocity for the total trip, (b) the average velocity during the first 4.0 s of motion, (c) the average velocity during the next 4.0 s of motion, (d) the instantaneous velocity at $t = 2.0$ s, and (e) the instantaneous velocity at $t = 9.0$ s.

Solution: (a) To find the average velocity of the entire trip, we draw a line connecting the origin to the final position (C) and find the slope of this line. We can also use the mathematical formula we've been given,

$$\begin{aligned}\overline{v} &= \frac{x_f - x_i}{t_f - t_i}\\ \overline{v} &= \frac{10\text{ m} - 0\text{ m}}{12\text{ s} - 0\text{ s}}\\ \overline{v} &= +0.83\text{ m/s}\end{aligned}$$

(b) For the first four seconds, we use the same idea, but the second endpoint is now at $t = 4.0$ s,

$$\begin{aligned}\overline{v} &= \frac{x_f - x_i}{t_f - t_i}\\ \overline{v} &= \frac{4.0\text{ m} - 0\text{ m}}{4.0\text{ s} - 0\text{ s}}\\ \overline{v} &= +1.0\text{ m/s}\end{aligned}$$

(c) For the next four seconds, we have to move both the start and endpoints,

$$\begin{aligned}\overline{v} &= \frac{x_f - x_i}{t_f - t_i}\\ \overline{v} &= \frac{4.0\text{ m} - 4.0\text{ m}}{8.0\text{ s} - 4.0\text{ s}}\\ \overline{v} &= +0\text{ m/s}\end{aligned}$$

(d) The instantaneous velocity is the limit of the average velocity at a specific time. The graph around $t = 2.0$ s is a straight line, which has a constant slope, so the instantaneous velocity is the same as the average velocity for that segment.

$$v = 1.0\text{ m/s}.$$

(e) At $t = 9.0$ s, the curve is not a straight line and the slope is changing around $t = 9.0$ s. To find the instantaneous velocity, we need to draw a line tangent to the curve (remember that the slope of this line gives us the instantaneous velocity). The tangent line crosses the x-axis at roughly (3.0 s,0 m) and hits the curve at (9.0 s,4.5 m). We can use these two points to find the slope of the line

$$\begin{aligned}v &= \frac{x_f - x_i}{t_f - t_i}\\ v &= \frac{4.5\text{ m} - 0\text{ m}}{9.0\text{ s} - 3.0\text{ s}}\\ v &= +0.75\text{ m/s}\end{aligned}$$

2.3 Acceleration

Objects rarely travel with constant velocity. When driving in your car, you regularly speed up or slow down. Your velocity will even change when you're making a turn — in this case you're changing the direction of velocity even if the magnitude stays the same. The change of an object's velocity is called *acceleration*.

We can define an *average acceleration* just as we defined an average velocity,

$$\overline{a} \equiv \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}, \tag{2.5}$$

where v_f is the final velocity and v_i is the initial velocity. Just like velocity and displacement, acceleration can be negative or positive with the sign telling us the direction of the acceleration, so acceleration is also a vector quantity. Note that a negative velocity does not necessarily mean that an object is slowing down. To determine whether an object is speeding up or slowing down, we need to consider the directions of both velocity and acceleration. If velocity and acceleration are in the same direction (both positive or both negative), then the object is speeding up. If velocity and acceleration are in opposite directions (one positive, one negative) then the object is slowing down.

2.3.1 Instantaneous acceleration

When we discussed velocity, we found it useful to plot the object's position versus time because the slope of that curve gives us the velocity of the object. For acceleration, we want to plot the object's velocity versus time. When we connect the two endpoints of a velocity vs. time curve, the slope of the resulting line gives us the object's acceleration. We can extend this idea, just as we did for velocity, to define an *instantaneous acceleration* by taking the limit as the endpoints get closer and closer,

$$a \equiv \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}. \tag{2.6}$$

Just as velocity is the slope of the line tangent to the position vs. time graph, we have that acceleration is the slope of the line tangent to the velocity vs. time graph.

Example: Passing out

The average person passes out at an acceleration of $7g$ (that is, seven times the gravitational acceleration on Earth). Suppose a car is designed to accelerate at this rate. How much time would be required for the car to accelerate from rest to 60.0 mph?

Solution: You may not have encountered the constant g before. It's value can be looked up in the Table on the inside back cover of your textbook and is $g = 9.80$ m/s^2. Now let's assess the problem. You are given an acceleration ($7g$) and a final velocity (60.0 mph) as well as an initial velocity (rest=0 mph). You want to find a time interval (Δt). The equation that defines average acceleration uses these quantities, so let's see if we can manipulate that to get what we want,

$$\begin{aligned}\overline{a} &= \frac{v_f - v_i}{\Delta t} \\ \Delta t &= \frac{v_f - v_i}{\overline{a}}.\end{aligned}$$

Now we have an expression for Δt in terms of quantities that we know. Before we go ahead and plug in the numbers, let's take a close look at the numbers we have: we have acceleration in units of m/s^2 and velocity in units of mph. **These will not cancel to give an answer in units of seconds!** We need to convert mph to m/s,

$$60 \text{ mph} \times \frac{1609 \text{ m}}{1 \text{ mi}} \times \frac{1 \text{ min}}{60 \text{ s}} \times \frac{1 \text{ h}}{60 \text{ min}} = 26.8 \text{ m/s}.$$

Now we can substitute into our equation,

$$\begin{aligned}\Delta t &= \frac{26.8 \text{ m/s} - 0}{7 \times 9.80 \text{ m/s}^2} \\ \Delta t &= 0.39 \text{ s}.\end{aligned}$$

It would only take 0.39 s for the car to reach 60 mph and for you to pass out.

2.4 1-D motion with constant acceleration

Many applications in physics involve motion with constant acceleration. In particular, an object falling due to the gravitational pull of a planet (usually Earth) falls with a constant acceleration if we ignore air resistance. Constant acceleration is very nice because the instantaneous acceleration at any point in time is equal to the average acceleration over the entire time period. This means that velocity increases or decreases at a constant rate throughout the entire period of motion, so the plot of velocity vs. time is a straight line with a slope given by the acceleration. Because the average acceleration and instantaneous acceleration are the same in this case, we will stop writing the bar over $\overline{a}$ and simply use a to denote acceleration. We can use our definition for average acceleration,

$$a = \frac{v_f - v_i}{t_f - t_i}, \tag{2.7}$$

to find the acceleration at any time during the motion. Since we can start measuring time at any time we wish, let's assume that the motion starts at $t_i = 0$, and we'll simply call $t_f = t$ (drop the f). We'll also re-label $v_i = v_0$ and $v_f = v$, then we can re-write the equation as

$$v = v_0 + at. \tag{2.8}$$

This equation tells us that the acceleration steadily changes the initial velocity v_0 by an amount that depends on how long we've been accelerating at. Because the velocity is changing linearly with time, we can calculate the average velocity using the initial and final velocities (before, we used initial and final position). The average velocity is the arithmetic average of the initial and final velocity,

$$\overline{v} = \frac{v_0 + v}{2}. \tag{2.9}$$

Remember that you can only use this equation when acceleration is constant, **it is not true otherwise**.

We also have an equation that defines average velocity and is true in all cases,

$$\overline{v} = \frac{\Delta x}{t}. \tag{2.10}$$

Let's solve this equation for Δx,

$$\Delta x = \overline{v}t, \tag{2.11}$$

and use the new equation for $\overline{v}$,

$$\begin{aligned} \Delta x &= \left(\frac{v_0 + v}{2}\right) t \\ \Delta x &= \frac{1}{2}(v_0 + v)t, \end{aligned} \tag{2.12}$$

to get an expression for displacement in terms of initial and final velocities. We can get an even more useful relationship by eliminating the final velocity. If we use Eq. (2.8) to substitute for the final velocity,

$$\begin{aligned} \Delta x &= \frac{1}{2}(v_0 + (v_0 + at))t \\ \Delta x &= v_0 t + \frac{1}{2}at^2. \end{aligned} \tag{2.13}$$

This expression is often handy because it does not contain the final velocity. In many cases, we have information about the start of motion, but we rarely have information about the end of it (that's usually what we are trying to predict).

We can get another useful relationship by solving Eq. (2.8) for time,

$$t = \frac{v - v_0}{a}, \tag{2.14}$$

and substituting into Eq. (2.12),

$$\begin{aligned} \Delta x &= \frac{1}{2}(v_0 + v)\left(\frac{v - v_0}{a}\right) \\ \Delta x &= \frac{v^2 - v_0^2}{2a} \\ v^2 &= v_0^2 + 2a\Delta x. \end{aligned}$$

This equation is very handy if information about time is not given in the problem; it still allows us to calculate the final velocity without knowing how long the object was accelerating.

Example: Car Chase

A car traveling at a constant speed of 24.0 m/s passes a trooped hidden behind a billboard. One second after the speeding car passes the billboard, the trooper sets off in chase with a constant acceleration of 3.00 m/s^2. (a) How long does it take the trooper to overtake the speeding car? (b) How fast is the trooper going at that time?

Solution: We first need to be clear about how we are measuring time and position in this problem. Let's measure them both relative to the trooper's motion. That is, $x = 0$ at the billboard where the trooper starts moving and $t = 0$ when the trooper starts moving. This means that the speeding car passed the trooper at $t = -1$ s.
(a) Now we need to think about what the problem is asking us to find. It asks for a time ("how long") at which the trooper overtakes the speeding car. At the exact moment that the trooper overtakes the car, their positions on our axis will be equal, i.e. $x_t = x_c$. So, we want to find the time at which the positions are equal.
We have just derived an equation which gives position as a function of time for constant acceleration,

$$\Delta x = v_0 t + \frac{1}{2}at^2.$$

First check that we can use this equation; are both objects undergoing constant acceleration? Yes, the problem says the trooper has a constant acceleration and the car has constant velocity (a constant acceleration of 0). So let's figure out the position of the trooper. The trooper's initial position is $x_{0t} = 0$, his initial velocity is $v_{0t} = 0$, and his acceleration is $a = 3.00$ m/s^2, so we have

$$x_t = \frac{1}{2}at^2.$$

Now we do the same for the car. It's initial position is $x_{0c} = 24.0$ m (it had a 1 s head start), it's initial velocity is $v_{0c} = 24.0$ m/s, and it's acceleration is $a_c = 0$, so we have

$$\begin{aligned} x_c - x_{0c} &= v_{0c}t \\ x_c &= x_{0c} + v_{0c}t. \end{aligned}$$

To find the time at which positions are the same, we set the two expressions equal and solve for time,

$$\begin{aligned} x_t &= x_c \\ \frac{1}{2}at^2 &= x_{0c} + v_{0c}t \\ \frac{1}{2}at^2 - v_{0c}t - x_{0c} &= 0. \end{aligned}$$

You get a quadratic expression for t, so you will need to use the quadratic formula (it's in your textbook),

$$\begin{aligned} t &= \frac{v_{0c} \pm \sqrt{v_{0c}^2 - 4\left(\frac{1}{2}a\right)(-x_{0c})}}{2\left(\frac{1}{2}a\right)} \\ t &= \frac{24.0 \text{ m/s} \pm \sqrt{(24.0 \text{ m/s})^2 - 4\left(\frac{1}{2} \cdot 3.00 \text{ m/s}^2\right)(-24.0 \text{ m})}}{2\left(\frac{1}{2} \cdot 3.00 \text{ m/s}^2\right)} \\ t &= 16.9 \text{ s}. \end{aligned}$$

There is also a negative root, but since we know that the trooper could not overtake the car before it passed him, this root is not physically meaningful.

(b) The trooper's speed at that time is a fairly straightforward problem. We have an equation that gives us final velocity if we know the acceleration, time and initial velocity (and we know all those now).

$$\begin{aligned} v &= v_0 + at \\ v &= (0) + 3.00\ \text{m/s}^2 (16.9\ \text{s}) \\ v &= 50.7\ \text{m/s}. \end{aligned}$$

As mentioned earlier, constant acceleration is particularly useful because objects moving due to the gravitational pull of an object will move with a constant acceleration if air resistance is neglected. Objects moving under the influence of gravity and without air resistance are said to be in *free fall*. Note that this does not mean that the object necessarily started from rest ($v_0 = 0$). Objects can be moving upward, like when you throw a ball, and still be considered free-falling. The magnitude of the free-fall acceleration is denoted by g and has a value of 9.80 m/s^2 on Earth (although it varies slightly depending on latitude). The direction of g is always towards the large object creating the gravitational pull. Because g is a constant, we can use all of the equations we just derived for constant acceleration.

Example: Rookie Throw

A ball is thrown from the top of a building with an initial velocity of 20.0 m/s straight upward, at an initial height of 50.0 m above the ground. The ball just misses the edge of the roof on its way down. Determine (a) the time needed for the ball to reach its maximum height, (b) the maximum height, (c) the time needed for the ball to return to the height from which it was thrown and the velocity of the ball at that instant, (d) the time needed for the ball to reach the ground. Neglect air drag.

Solution: You can refer to the figure in your textbook (Fig. 2.20) to get a visual idea of what's going on in the problem. We are expressly given an initial velocity and an initial height in the problem. There are a few other pieces of information that are not expressly stated in the problem. The ball is in free fall, so we know the acceleration, $g = 9.80\ \text{m/s}^2$, and we know that the velocity of the ball at it's maximum height will be zero.

Let's set up our coordinate system so that $y = 0$ corresponds to the top of the building; this means that the bottom of the building is at -50.0 m.

(a) The first part of the problem is concerned with the upward motion of the ball. We are given an initial velocity, we know acceleration and a final velocity, and we want time. We have an equation that contains all these quantities,

$$\begin{aligned} v &= v_0 + at \\ t &= \frac{v - v_0}{a} \\ t &= \frac{0 - 20.0\ \text{m/s}}{-9.80\ \text{m/s}^2} \\ t &= 2.04\ \text{s}. \end{aligned}$$

(b) Now we want to know the maximum height reached by the ball. We have all the information from before and we now have the time at which we reach this maximum height, so we can use

$$\begin{aligned}
y &= y_0 + v_0 t + \frac{1}{2}at^2 \\
y_{max} &= (0) + (20.0\text{ m/s})(2.04\text{ s}) + \frac{1}{2}(-9.80\text{ m/s}^2)(2.04\text{ s})^2 \\
y_{max} &= 20.4\text{ m.}
\end{aligned}$$

(c) Now the ball begins to move downward and we want to know how long it takes to get back to it's initial height. We can use the same distance equation, but now use the fact that $y_f = 0$ to solve for time.

$$\begin{aligned}
y &= y_0 + v_0 t + \frac{1}{2}at^2 \\
v_0 t + \frac{1}{2}at^2 &= 0 \\
t(v_0 + \frac{1}{2}at) &= 0.
\end{aligned}$$

We will have two roots, $t = 0$ because the ball starts at that height, and the one we really want,

$$\begin{aligned}
v_0 + \frac{1}{2}at &= 0 \\
t &= -\frac{2v_0}{a} \\
t &= -\frac{2(20.0\text{ m/s})}{-9.80\text{ m/s}^2} \\
t &= 4.08\text{ s.}
\end{aligned}$$

Note that this is twice the time it takes to get to the maximum height. The velocity of the ball at that time is

$$\begin{aligned}
v &= v_0 + at \\
v &= 20.0\text{ m/s} + (-9.80\text{ m/s}^2)(4.08\text{ s}) \\
v &= -20.0\text{ m/s.}
\end{aligned}$$

(d) Now our final position is $y_f = -50.0$ m, but this is essentially the same problem we just solved.

$$\begin{aligned}
y &= y_0 + v_0 t + \frac{1}{2}at^2 \\
\frac{1}{2}at^2 + v_0 t - y_f &= 0.
\end{aligned}$$

This leads to a quadratic equation, so it's a little more effort to solve,

$$\begin{aligned}
t &= \frac{v_0 \pm \sqrt{v_0^2 - 4\left(\frac{1}{2}a\right)(-y_f)}}{2\left(\frac{1}{2}a\right)} \\
t &= \frac{20.0\text{ m/s} \pm \sqrt{(20.0\text{ m/s})^2 - 4\left(-\frac{1}{2}\cdot 9.80\text{ m/s}^2\right)(-(-50.0\text{ m}))}}{2\left(\frac{1}{2}\cdot 9.80\text{ m/s}^2\right)} \\
t &= 5.83\text{ s.}
\end{aligned}$$

Chapter 3

Vectors and Two-Dimensional Motion

We live in a three-dimensional world and the objects around us move in that three-dimensional space. While some simple examples of motion can be described in one dimension, extend what we've learned about motion to more than one dimension will allow us to study many more systems.

3.1 Vector properties

Displacement, velocity and acceleration are all vector quantities. That is, they have a magnitude and a direction. In one dimension, there were two options for the direction, left or right, and we could represent the direction of displacement, velocity, or acceleration simply by using a negative or positive sign. When we move to two dimensions, there are many more options for the direction of the vector and we will need to be more formal with the mathematics of vectors.

Vectors are represented graphically as arrows with the length of the arrow representing the magnitude of the vector and the direction of the arrow giving the direction of the vector. This graphical representation might help you understand the basic arithmetic of vectors. Mathematically, we denote a vector, $\vec{A}$, with an arrow over the variable name. If we use the variable name without the arrow, this means we are referring to the magnitude of the vector.

Let's set up a coordinate system at the starting end of the vector (see Fig. 3.1). Then the vector points to a specific location in that space. We know that we can give the coordinates of that location using either Cartesian, (x, y), or polar, (r, θ) coordinates. The polar coordinates of this point are quite straightforward; r is the length (magnitude) of the vector and θ is the angle between the vector and the x-axis. We can also determine the x and y coordinates by finding the *projections* of the vector on the x- and y-axes. The projection of a vector $\vec{A}$ along the x-axis is called the x-component and is represented by A_x. The projection of $\vec{A}$ along the y-axis is called the y-component and is represented by A_y. We can find the x and y components by converting the polar coordinates to rectangular coordinates,

$$\begin{aligned} A_x &= A\cos\theta \\ A_y &= A\sin\theta. \end{aligned} \tag{3.1}$$

Note that we can go from the vector's x- and y-components back to the polar representation using the Pythagorean theorem and the definition of the tangent:

$$\begin{aligned} A^2 &= A_x^2 + A_y^2 \\ \tan\theta &= \frac{A_y}{A_x}. \end{aligned} \tag{3.2}$$

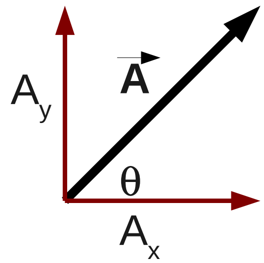


Figure 3.1: **The projections of a vector on the x- and y-axes.** Projections give the x- and y components of a vector.

Equality of vectors Two vectors are equal only if they have the same magnitude and direction (two arrows are the same only if they are the same length and point in the same direction). This means that you can move a vector around in space as long as you don't change the length or direction of the vector.

Adding vectors Just as when you are adding scalar quantities, you must ensure that the vectors you are trying to add have the same units. Vectors can be added either geometrically (graphically) or algebraically. To graphically add two vectors, $\vec{A}$ and $\vec{B}$, draw the vectors (using the same scale for both) head to tail on a piece of paper. The resultant vector $\vec{R} = \vec{A} + \vec{B}$ is the vector drawn from the unmatched tail to the unmatched head (see Fig. 3.2). You can lay out many vectors head to tail to find the result of them all. While graphical addition of vectors is useful for visualizing the addition process. You will most often be adding the vectors algebraically. To add vectors algebraically, resolve the vectors into their x- and y- components. All the x-components are added to get the x-component of the resultant vector. All the y-components are added to get the y-component of the resultant vector. **Never add the the x-components to y-components**. You can get the magnitude and direction of the resultant by converting from the Cartesian representation (x and y components) using the equations presented earlier.

Negative of a vector The negative of a vector, $\vec{A}$, is defined as the vector that gives zero when added to $\vec{A}$. If you think about the graphical addition of vectors, this means that the negative of $\vec{A}$ must have the same magnitude as $\vec{A}$, but opposite direction (180° difference).

Subtracting vectors Subtraction is simply the addition of a negative quantity. Since we have now defined the negative of a vector, we can figure out how to subtract. Remember that the negative of a vector points in the opposite direction, so for subtraction we flip the direction of the vector to be subtracted, but otherwise use the same graphical method as for addition. Graphically flipping a vector corresponds to algebraically changing the sign of both the x- and y- components of the vector, so we can use the same algebraic method for subtracting vectors as we do for adding vectors.

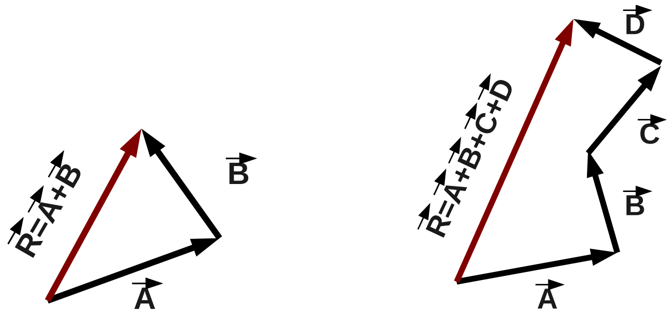


Figure 3.2: **Graphical addition of vectors.** To graphically add vectors, the vectors are drawn head to tail. The resultant vector is the vector that connects the two "loose ends", drawn from tail to head.

Example: Take a Hike

A hiker begins a trip by first walking 25.0 km 45.0° south of east from her base camp. On the second day, she walks 40.0 km in a direction 60.0° north of east, at which point she discovers a forest ranger's tower. (a) Determine the components of the hiker's displacements in the first and second days. (b) Determine the components of the hiker's total displacement for the trip. (c) Find the magnitude and direction of the displacement from base camp.

Solution: (a) Let's set the origin of our coordinate system at the camp and have the x-axis pointing east and the y-axis pointing north. On the first day, the hiker's displacement, let's call it $\vec{A}$ has a magnitude of 25.0 km with a direction $\theta = -45°$ — it's negative because she is moving south of the x-axis. We can use Eqs. 3.1 to find the x- and y-components,

$$\begin{aligned} A_x &= A\cos\theta = (25\text{ km})\cos(-45°) = 17.7\text{ km} \\ A_y &= A\sin\theta = (25\text{ km})\sin(-45°) = -17.7\text{ km}. \end{aligned}$$

On the second day, her displacement, let's call it $\vec{B}$, has a magnitude of 40.0 km with a direction $\theta = 60.0°$ — positive this time because it is north of east. The x- and y-components of this displacement are

$$\begin{aligned} B_x &= B\cos\theta = (40\text{ km})\cos(60°) = 20.0\text{ km} \\ B_y &= B\sin\theta = (40\text{ km})\sin(60°) = -34.6\text{ km}. \end{aligned}$$

(b) The total displacement is the vector sum of $\vec{A}$ and $\vec{B}$. We've just found the components of $\vec{A}$ and $\vec{B}$, so we can find the components of the total displacement,

$$\begin{aligned} R_x &= A_x + B_x = 17.7\text{ km} + 20.0\text{ km} = 37.7\text{ km} \\ R_y &= A_y + B_y = -17.7\text{ km} + 34.6\text{ km} = 16.9\text{ km}. \end{aligned}$$

(c) We just found the components of the total displacement, all we need to do is convert them to a magnitude and direction. The magnitude is found using the Pythagorean theorem,

$$\begin{aligned} R^2 &= R_x^2 + R_y^2 \\ R &= \sqrt{(37.7 \text{ km})^2 + (16.9 \text{ km})^2} \\ R &= 41.3 \text{ km}. \end{aligned}$$

The direction is found using the tangent function,

$$\begin{aligned} \tan\theta &= \frac{R_y}{R_x} \\ \tan\theta &= \frac{16.9 \text{ km}}{37.7 \text{ km}} \\ \theta &= \tan^{-1}\left(\frac{16.9 \text{ km}}{37.7 \text{ km}}\right) \\ \theta &= 24.1^\circ. \end{aligned}$$

Don't forget to check that the angle is in the correct quadrant. In this case, the x- and y-components are both positive, so we are in the first quadrant and the angle is correct.

3.1.1 Displacement, velocity and acceleration in two dimensions

In one dimension, we defined the displacement as the difference between the initial and final positions of an object. The position in that case was determined by a single coordinate. We now want to define displacement in two dimensions where the position of the object is given by two coordinates. Let's call the initial position of the object $\vec{r_i}$ and the final position of the object $\vec{r_f}$, where $\vec{r}$ is a *position vector* that goes from the origin to the position of the object. The displacement is defined as the vector difference between the initial and final position vectors,

$$\Delta\vec{r} = \vec{r_f} - \vec{r_i}. \tag{3.3}$$

With this generalized definition of displacement, we can also generalize the average velocity and average acceleration of an object,

$$\vec{v_{av}} = \frac{\Delta\vec{r}}{\Delta t} \tag{3.4}$$

$$\vec{a_{av}} = \frac{\Delta\vec{v}}{\Delta t}. \tag{3.5}$$

Finally, we can also generalize the instantaneous velocity and instantaneous acceleration

$$\vec{v} = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} \tag{3.6}$$

$$\vec{a} = \lim_{\Delta t\to 0}\frac{\Delta\vec{v}}{\Delta t}. \tag{3.7}$$

If you look carefully at these definitions, you will see that x-component of acceleration is determined by the x-component of velocity which is determined by the x-component of the displacement. The same is true for the y-components of these quantities. This means that the horizontal and vertical components can be treated independently of each other.

Example: The swimmer

The current in a river is 1.0 m/s. A woman swims across the river to the opposite bank and back. She can swim 2.0 m/s in still water and the river is 300 m wide. She swims perpendicular to the current so she ends up downstream from where she started. Find the time for the round trip

Solution: Since the woman swims perpendicular to the current let's define the y-axis as parallel to the river. We can treat the x and y motion independently. We are only interested in the motion in the x-direction (across the river) since this will determine how long the trip takes. She is swimming at a constant velocity of 2.0 m/s, so the time to travel a distance of 300 m is

$$\begin{aligned}
\overline{v_x} &= \frac{\Delta x}{t} \\
t &= \frac{\Delta x}{\overline{v_x}} \\
t &= \frac{300 \text{ m}}{2.0 \text{ m/s}} \\
t &= 150 \text{ s.}
\end{aligned}$$

It will take the same amount of time for her to travel back, so the round trip takes 300 s. Note that the current pushing the woman down the river is completely irrelevant here because it affects her motion in the y-direction (along the river) and this is independent of her motion in the x-direction.

3.2 Motion in two dimensions

We have previously studied motion of objects moving in a straight line (one dimension). We will now extend our study to two dimensions. We know that if we break motion up into x and y components, that the motion in the two directions is independent, so that motion in the horizontal direction does not affect motion in the vertical direction and vice versa. This is particularly important when studying something called *projectile motion* which is the motion of any object thrown in some way. If we throw a ball with some horizontal initial velocity, its motion can be studied by breaking it up into the horizontal and vertical motions. In the vertical direction, the object undergoes acceleration due to gravity just as in free fall. In the horizontal direction, there is no acceleration and the velocity remains constant. The resulting two-dimensional motion is the combination of the two components.

Chapter 4

Laws of motion

So far, we've studied motion by describing what happens without being concerned about what causes the motion. Now, we will start to examine the causes of motion and we will learn the rules that govern changes in motion.

4.1 Newton's first law

Isaac Newton developed the laws of motion in the 1600s when he started thinking about why objects close to the Earth tended to fall to Earth unless something was holding them up in some way. His ideas on motion are summed up in three laws that are based on the idea of forces.

You probably have an intuitive sense of a force from everyday life. When you push or pull on an object, you are applying an external force on the object. These are examples of *contact forces*, forces which are caused by one object being in contact with another object. There are also forces that arise without contact of two objects. While this may seem strange (Newton was also uncomfortable with the idea of action-at-a-distance), you are very familiar with one such force. Gravity causes all objects near Earth to fall towards the Earth even though the Earth is not touching the object. This is an example of a *field force*, so called because scientists use the idea of a force field emanating from an object to explain how it might affect the motion of objects that it hasn't touched. Essentially a force is something that can change the state of motion of an object. Note that force (contact or field) is a vector — it has both a magnitude and direction.

If a force is something that can change the state of motion of an object, will objects move without a force? Obviously, if an object is at rest (not moving), it will just sit there forever unless something pushes or pulls it. Suppose now that the object is given a quick push. It will start moving because of the force that has been applied, but what happens after the initial push? In most cases, the object will start to slow down because there is friction between it and the object on which it moves. But suppose we could eliminate the friction, which is a force and changes the motion of the object? If we completely eliminate friction, then the object would continue moving without speeding up or slowing down. So yes, objects will move without the presence of a force, but with a very specific type of motion. This is the essence of Newton's first law,

> "An object moves with a velocity that is constant in magnitude and direction unless a non-zero net force acts on it."

The net force is the vector sum of all the forces acting on the object. So this law can be used in two ways. If we know that there is no net force on the object, then we know that it will continue moving with a constant (possibly zero) velocity. Alternatively, if an object is moving with a constant (possibly zero) velocity, then the net force acting on it must be zero.

This law is based on the notion of *inertia* which is the tendency of an object to continue its state of motion in the absence of a force. The law is sometimes stated as "a body in motion will stay in motion and a body at rest will stay at rest unless acted upon by an outside force." This is closely related to the idea of *mass*, which measures an object's resistance to changes in its velocity due to a force. If the same force acts on two objects with different masses, the object with the smaller mass will experience a bigger change in its velocity than the more massive object.

4.2 Newton's second law

In the absence of a force, an object will keep doing whatever it was doing. What happens if a force acts on the object? Clearly it's velocity will change in some way. If you push on an object, it will accelerate (change its velocity). If you push harder, it will accelerate faster. So the magnitude of the force is related to the acceleration. In fact, the force is proportional to the acceleration; so if you push twice as hard the acceleration will be twice as large.

What other things might affect the acceleration? One quantity that we've already discussed is the mass. The same force applied to objects with different masses will result in different accelerations. In this case the relationship is inversely proportional — if the mass is twice as large, the acceleration is halved.

Newton put both of these observations together into his second law

> "The acceleration $\vec{a}$ of an object is directly proportional to the net force acting on it and inversely proportional to its mass."

We can write this statement more compactly using mathematics

$$\vec{a} = \frac{\sum \vec{F}}{m}, \tag{4.1}$$

where $\vec{a}$ is the acceleration (vector) of the object, $\sum \vec{F}$ is the vector sum of of all the forces acting on the object and m is the mass of the object. To actually use this equation, we break it up into its x and y (and maybe z) components

$$\begin{aligned} \sum F_x &= ma_x \\ \sum F_y &= ma_y. \end{aligned} \tag{4.2}$$

Note that the unit of force in the SI system is the **newton** where $1\ \text{N} = 1\ \text{kg} \cdot \text{m/s}^2$.

Example: Horses pulling a barge

Two horses are pulling a barge with mass 2.0×10^3 kg along a canal. The cable connected to the first horse makes an angle of $\theta_1 = 30.0°$ with respect to the direction of the canal, while the cable connected to the second horse makes an angle of $\theta_2 = -45.0°$. Find the initial acceleration of the barge, starting at rest, if each horse exerts a force of magnitude 6.00×10^2 N on the barge. Ignore forces of resistance on the barge.

Solution: We've been given the mass of an object and the forces acting on it and we're asked to find accelerations. So we want to use Newton's second law to try to find the acceleration. Let's define our coordinate system with the x-axis lying along the canal (so the angles are measured relative to the x-axis). Now we can break down the forces into x and y components,

$$\begin{aligned}
F_{1x} &= F\cos\theta_1 = (6.00 \times 10^2\ \text{N})\cos(30.0°) = 5.2 \times 10^2\ \text{N} \\
F_{1y} &= F\sin\theta_1 = (6.00 \times 10^2\ \text{N})\sin(30.0°) = 3.00 \times 10^2\ \text{N} \\
F_{2x} &= F\cos\theta_2 = (6.00 \times 10^2\ \text{N})\cos(-45.0°) = 4.24 \times 10^2\ \text{N} \\
F_{2y} &= F\sin\theta_2 = (6.00 \times 10^2\ \text{N})\sin(-45.0°) = -4.24 \times 10^2\ \text{N}.
\end{aligned}$$

Newton's second law tells us to find the *net* force in both the x and y directions,

$$\begin{aligned}
F_x &= F_{1x} + F_{2x} = 5.2 \times 10^2\ \text{N} + 4.24 \times 10^2\ \text{N} = 9.44 \times 10^2\ \text{N} \\
F_y &= F_{1y} + F_{2y} = 3.00 \times 10^2\ \text{N} - 4.24 \times 10^2\ \text{N} = -1.24 \times 10^2\ \text{N}.
\end{aligned}$$

Now Newton's second law says that the net force is related to the acceleration,

$$\begin{aligned}
a_x &= \frac{F_x}{m} = \frac{9.44 \times 10^2\ \text{N}}{2.0 \times 10^3\ \text{kg}} = 0.472\ \text{m/s}^2 \\
a_y &= \frac{F_y}{m} = \frac{-1.24 \times 10^2\ \text{N}}{2.0 \times 10^3\ \text{kg}} = -0.062\ \text{m/s}^2.
\end{aligned}$$

We have the x and y components of the acceleration, so we can find the magnitude and acceleration,

$$a = \sqrt{a_x^2 + a_y^2} = 0.476\ \text{m/s}^2$$

$$\theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) = -7.46°.$$

4.2.1 Weight

The weight of an object is not the same as its mass. The weight of an object is the magnitude of the gravitational force acting on an object, so on Earth the weight is $F_w = mg$. Because g is the same everywhere on Earth, we can use an object's weight to determine its mass and so the two terms are used interchangeably in everyday language. While mass is a fundamental property of an object and will not change if you move the object to another location, its weight can change depending on the object's location.

4.3 Newton's third law

Newton's third law is perhaps the least intuitive of the three laws of motion. According to Newton, all forces come in pairs, "If object 1and object 2 interact, the force $\vec{F_{12}}$ exerted by object 1 on object 2 is equal in magnitude but opposite in direction to the force $\vec{F_{21}}$ exerted by object 2 on object 1"

The force exerted by object 1 on object 2 is sometimes called the *action* force and the force exerted on object 2 by object 1 is called the *reaction* force. The law is sometimes stated as "every action has an equal and opposite reaction". Basically anytime two objects interact in some way, there will be **two** forces, one acting on each object. When you walk, your foot pushes on the floor and the floor pushes back on you. When you lean against a wall, the wall pushes back on you. Every time an object falls towards Earth because Earth's gravity is pulling it, the object also pulls Earth towards it. This may seem strange, but the Earth is so much more massive than other objects that its acceleration due to this force is negligible.

One consequence of this law is a force called the *normal force*. Every object on Earth is being pulled towards the center of the Earth by gravity. Most objects are not moving downward because they are sitting on some surface and this surface is pushing up on the object. The upwards force is the normal force, so named because it is **always perpendicular to the surface**; this mean it does not always point straight up even though it is a reaction force to the pull of gravity.

Example: Standing on a crate

A 38 kg crate rests on a horizontal floor, and a 63 kg person is standing on the crate. (a) Determine the magnitude of the normal force that the crate exerts on the person. (b) Determine the magnitude of the normal force that the floor exerts on the crate.

Solution: (a) Let's consider the forces acting on the person only. There is, of course a downward force due to gravity $F_g = m_p g$. There is also a normal force from the crate pushing up on the person. The man is not moving so these two forces must be in equilibrium (the net force must be zero),

$$\begin{aligned} -F_g + F_{cp} &= 0 \\ F_{cp} &= m_p g. \end{aligned}$$

(b) Now let's look at the forces acting on the crate. Gravity acts on the crate, $F_g = m_c g$, and the floor pushes up on the crate through the normal force, F_{fc}. The person standing on the crate also pushes down on the crate with a "normal" force that is equal in magnitude and opposite to the force of the crate pushing on the person, $F_{pc} = -F_{cp}$. The crate is not moving,so these forces must be in equilibrium,

$$\begin{aligned} F_{fc} - F_g - F_{pc} &= 0 \\ F_{fc} &= m_c g + m_p g. \end{aligned}$$

When we are using Newton's laws to solve problems, we use several assumptions. We assume that each object is a *point mass*, or that they are particles without any spatial extent (0-dimensional objects). This means we don't have to worry about rotation of the objects. If strings or ropes are part of the problem, we assume that their mass is negligible and that any *tension* in the rope is the same at all points on the rope.

When solving force problems, it is useful to draw a *free-body diagram*. The free body diagram is a drawing of all the forces acting on a particular object. It is very important to **only draw the forces *acting on* the object**, any force that the object exerts on its surroundings is not included in the free-body diagram. This diagram helps to isolate the forces of interest for our object and can then be used to apply Newton's laws.

Example: Traffic light

A traffic light weighing 1.00×10^2 N hangs from a vertical cable tied to two other cables that are fastened to a support. The upper cables make angles of 37° and 53° with the horizontal (see Fig. 4.14 in your textbook). Find the tension in each of the three cables.

Solution: We start by drawing free-body diagrams. First for the traffic light which has gravity acting downward (the weight, $\vec{W}$ and tension, $\vec{T_3}$ from the rope pulling it upward. Because both forces are in the y direction only, this leads to a single equation,

$$T_3 - W = 0.$$

Although we now know the value of T_3, this does not tell us anything about the tension in the other two ropes. We can also consider the forces acting on the knot. The knot has three ropes pulling on it: tension downwards from $\vec{T_3}$, tension up to the left from $\vec{T_1}$ and tension up to the right from $\vec{T_2}$. Because forces are vectors, we need to break everything into x and y components and apply Newton's second law along each axis. Note that the knot is in equilibrium, so there is no acceleration in either direction,

$$\begin{aligned}
\text{x} - \text{direction}: &\quad T_2 \cos(53^\circ) - T_1 \cos(37^\circ) = 0 \\
\text{y} - \text{direction}: &\quad T_1 \sin(37^\circ) + T_2 \sin(53^\circ) - T_3 = 0.
\end{aligned}$$

We know T_3 from the first equation, so we are left T_1 and T_2 as unknowns. Luckily, we have two equations for our two unknowns. so we can solve one equation,

$$T_2 = T_1 \frac{\cos(37^\circ)}{\cos(53^\circ)}$$

and substitute into the other equation,

$$\begin{aligned}
T_1 \sin(37^\circ) + T_1 \frac{\cos(37^\circ)}{\cos(53^\circ)} \sin(53^\circ) - W &= 0 \\
T_1 (\sin(37^\circ) + \frac{\cos(37^\circ)}{\cos(53^\circ)} \sin(53^\circ)) &= W \\
T_1 &= \frac{W}{\sin(37^\circ) + \frac{\cos(37^\circ)}{\cos(53^\circ)} \sin(53^\circ)} \\
T_1 &= 60.1 \text{ N}.
\end{aligned}$$

We can now find T_2 as well,

$$\begin{aligned}
T_2 &= (60.1 \text{ N}) \frac{\cos(37^\circ)}{\cos(53^\circ)} \\
T_2 &= 79.9 \text{ N}.
\end{aligned}$$

4.4 Friction

An object moving on a surface or through some medium encounters resistance as it moves. This resistance is called *friction*. Friction is an essential force as it allows us to hold objects, drive a car and walk. The strength of the frictional force depends on whether an object is stationary or moving. You've undoubtedly had the experience of trying to push a large heavy object; it takes more effort to get the object moving than to keep it moving once it's going. When an object is stationary, the frictional force is called the force of *static friction* and when the object is moving, the frictional force is called the force of *kinetic* friction. Friction points opposite to the direction of motion in the kinetic case or the direction of impending motion in the static case. The force of static friction is larger than the force of kinetic friction.

It has been shown experimentally that both kinetic and static friction are proportional to the normal force. The only way this can be true is if the two forces have different constants of proportionality. For static friction, we have

$$f_s \leq \mu_s N \tag{4.3}$$

where μ_s is the coefficient of static friction. This is an inequality because the force of static friction can take on smaller values if less force is needed to hold an object in place. You will almost always use the '=' sign in problems. For the force of kinetic friction, we have

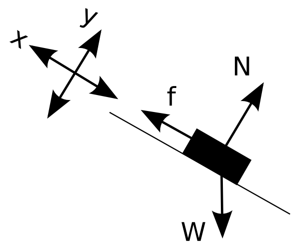
$$f_k = \mu_k N \tag{4.4}$$

where μ_k is the coefficient of kinetic friction. μ_k will almost always be less than μ_s. Note that the friction coefficients do not have any units.

Example: Block on a ramp

Suppose a block with a mass of 2.50 kg is resting on a ramp. If the coefficient of static friction between the block and ramp is 0.350, what maximum angle can the ramp make with the horizontal before the block starts to slip down?

Solution: This is a problem involving forces, so we will need a free body diagram.



There are three forces acting on the block: gravity pulls straight down, the normal force acts perpendicular to the surface, and the force of friction acts upwards along the ramp because the block would like to slide down the ramp. We will choose a tilted coordinate system such that the x-axis runs along the ramp. If we do this, the only force that needs to be broken into components is the force of gravity (friction pulls entirely along the x axis and the normal force pulls entirely along the y axis). We can now use Newton's second law for both axes to get two equations,

$$\begin{aligned} \sum F_x &= 0 \\ mg\sin\theta - f &= 0 \end{aligned}$$

$$\begin{aligned} \sum F_y &= 0 \\ N - mg\cos\theta &= 0. \end{aligned}$$

Remember that the force of friction is related to the normal force,

$$f = \mu_s N.$$

I have used the equal sign here because the static force of friction will be largest (and therefore equal) just before the object starts to slip. Let's use one equation to solve for N,

$$N = mg\cos\theta$$

and substitute into the other equation,

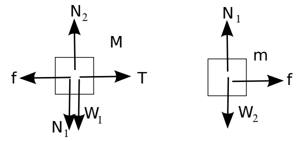
$$\begin{aligned} mg\sin\theta - \mu_s mg\cos\theta &= 0 \\ \tan\theta &= \mu_s \\ \theta &= 19.3^\circ. \end{aligned}$$

The angle at which the block will slide depends only on the coefficient of static friction between the ramp and block.

Example: Two blocks

A block of mass $m = 5.00$ kg rides on top of a second block of mass $M = 10.0$ kg. A person attaches a string to the bottom block and pulls the system horizontally across a frictionless surface. Friction between the two blocks keeps the 5.0 kg block from slipping off. If the coefficient of static friction is 0.305, (a) what maximum force can be exerted by the string on the 10.0 kg block without causing the 5.0 kg block to slip? (b) What is the acceleration?

Solution: This is a problem involving forces, so we will need a free body diagram.



There are five forces acting on block M: gravity pulls straight down, the normal force from the floor pushes up, the normal force from block m pushes down, the tension from the rope, and friction from block m opposes the motion (so f points to the left). There are three forces acting on block m: the normal force from block M pushing up, gravity pulling down and the force of friction that opposes the potential motion of block m (so f points to the right). (a) Now let's use Newton's second law on both objects. First block M,

$$\begin{aligned} \sum F_x &= Ma \\ T - f &= Ma \end{aligned}$$

$$\begin{aligned} \sum F_y &= 0 \\ N_2 - N_1 - Mg &= 0, \end{aligned}$$

and for the second block,

$$\begin{aligned} \sum F_x &= ma \\ f &= ma \end{aligned}$$

$$\begin{aligned} \sum F_y &= 0 \\ N_1 - mg &= 0, \end{aligned}$$

Remember that the force of friction is related to the normal force,

$$f = \mu_s N_1.$$

I have used the equal sign here because the static force of friction will be largest (and therefore equal) just before the object starts to slip. Now we want to find T,

$$N_1 = mg$$

$$\begin{aligned} \mu_s N_1 &= ma \\ a &= \mu_s g \end{aligned}$$

$$\begin{aligned} T - f &= Ma \\ T - \mu_s mg &= M\mu_s g \\ T &= (m + M)\mu_s g \\ T &= 51.5 \text{ N} \end{aligned}$$

(b) We already have an expression for the acceleration

$$\begin{aligned} a &= \mu_s g \\ a &= 3.43 \text{ m/s}^2. \end{aligned}$$

Chapter 5

Work and Energy

5.1 Work

You probably define work as something that expends some of your energy. Typing up a paper, or writing out a homework assignment is considered work as are more physically demanding tasks such as building furniture or moving heavy objects. In physics, work has a very specific definition that involves motion and forces. Work is done by a force *only if* that force causes a net displacement of the object. There are two key points here: a force is only responsible for motion if it is in the same direction as that motion, so forces that are perpendicular to motion do not result in work, and there must be a net displacement or no work has been done.

The mathematical definition of work done on an object is

$$W = F\Delta x \cos\theta \tag{5.1}$$

where F is the force applied to the object, Δx is the displacement, θ is the angle between the force and the direction of motion, and W is the work. Work is measured in units of joules (joule). Note that if the force is doubled, work is doubled or if the object is displaced twice as far, then work is also doubled. Work is a scalar quantity — it has a magnitude, but no direction. Work can, however, be positive or negative; it's negative when the applied force is opposite to the direction of motion.

Example: Work on a block

A block of mass $m = 2.50$ kg is pushed a distance $d = 2.20$ m along a frictionless horizontal table by a constant applied force of magnitude $F = 16.0$ N directed at an angle $\theta = 25.0°$ below the horizontal. Determine the work done by (a) the applied force, (b) the normal force exerted by the table, (c) the force of gravity, and (d) the net force on the block.

Solution: (a) The applied force has both horizontal and vertical components, but because the motion is entirely horizontal, only the horizontal component contributes to the force.

$$\begin{aligned} W &= Fd\cos\theta \\ W &= (16.0\ \text{N})(2.20\ \text{m})\cos(-25.0°) \\ W &= 32\ \text{J}. \end{aligned}$$

(b) The normal force is perpendicular to the motion, so it does not do any work.
(c) The force of gravity is also perpendicular to the motion, so it also does no work.
(d) We could calculate the net force by vectorially adding the normal force, the applied force and the force of gravity and then find the work done by that force. Or we can save ourselves some effort by remembering that only forces along the direction of motion contribute to the work. The only force that has a horizontal component is the applied force, and we found the work due to that force in part (a).

5.2 Kinetic energy

Energy is an indirectly observed quantity that measures an object's capacity to do work. Energy comes in many different forms and can easily change from one form to another, but the total amount of energy in the universe (or in an isolated system) stays the same. That means that energy is a *conserved* quantity. The concept of energy provides an alternative formulation for Newton's laws. An object's energy determines it's potential to do work and the work it can do is related to the net force exerted by the object.

$$W_{\text{net}} = F_{\text{net}}\Delta x = ma\Delta x.$$

If the force is constant, the acceleration is also constant and we can use kinematics equations, namely,

$$\begin{aligned} v^2 - V_0^2 &= 2a\Delta x \\ a\Delta x &= \frac{v^2 - v_0^2}{2}. \end{aligned}$$

We can substitute this into the work equation,

$$\begin{aligned} W_{\text{net}} &= m\left(\frac{v^2 - v_0^2}{2}\right) \qquad (5.2) \\ W_{\text{net}} &= \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2. \end{aligned}$$

This equation tells us that the net work done on an object leads to a change in a quantity of the form $\frac{1}{2}mv^2$. This term is called the *kinetic energy* of the object and it is the energy of the motion of the object. The equation tells us that any net work done on an object leads to a change in its kinetic energy and for this reason, the equation is known as the work-energy theorem.

It's important to realize that this is just an alternative formulation of Newton's second law — two different ways of looking at the same process. Both Newton's second law and the work-energy theorem tell us that interacting with an object in a certain way (by applying a force in Newton's view, or doing work in the energy view) will lead to changes in the object's velocity. Why do we need two ways to describe the same process? It is sometimes more convenient to use one formulation over the other when solving problems. The energy formulation uses scalars rather than vectors, which can be easier for calculations, but it is sometimes hard to determine the net work done on an object without considering forces.

Example: Stopping a ship

A large cruise ship of mass 6.50×10^7 kg has a speed of 12.0 m/s at some instant. (a) What is the ship's kinetic energy at this time? (b) How much work is required to stop it? (c) What is the magnitude of the constant force required to stop it as it undergoes a displacement of 2.5 km?

Solution: (a) We know the ship's initial speed and we know that the ship's kinetic energy is determined by its speed,

$$\begin{aligned} K_0 &= \frac{1}{2}mv^2 \\ K_0 &= \frac{1}{2}(6.50 \times 10^7 \text{ kg})(12.0 \text{ m/s})^2 \\ K_0 &= 4.7 \times 10^9 \text{ J.} \end{aligned}$$

(b) We want the ship's final velocity (and also its kinetic energy) to be 0. The work-energy theorem tells us how much work is required to change an object's kinetic energy,

$$\begin{aligned} W &= K_f - K_0 \\ W &= -4.7 \times 10^9 \text{ J.} \end{aligned}$$

(c) Remember that work is done by a force acting on an object that travels some distance. The force slowing down the ship in this case is the drag or friction force of the water. Remember that friction is always opposite to the direction of motion so $\theta = 180°$.

$$\begin{aligned} W &= F\Delta x \cos\theta \\ F &= \frac{W}{\Delta x \cos\theta} \\ F &= \frac{-4.7 \times 10^9 \text{ J}}{2500 \text{ m} \cos(180°)} \\ F &= 1.9 \times 10^6 \text{ N.} \end{aligned}$$

5.2.1 Conservative and nonconservative forces

Forces can be broken up into two types: conservative and non-conservative. Conservative forces are forces where you can easily get back the energy you put into system. Gravity is one example of a conservative force. If you lift a book, you will be doing work against gravity to raise that book. As you lower the book, the book is now doing work on you (the normal force still points the same way, but the motion is in the opposite direction, so work is negative) and you will recover all the energy you put into the book to lift it. A nonconservative force converts energy of objects into heat or sound — forms of energy that are hard to convert back to motion. Friction is one example of a nonconservative force — you can't recapture the energy lost to friction simply by moving the object back to where it started (like we did when we lowered the textbook).

The proper physics definition of a conservative force is

> "A force is conservative if the work it does moving an object between two points is the same no matter what path is taken."

This is based on the idea that, for conservative forces, we can get back the energy we put in simply by moving the object back to its starting point. We can re-write the work energy theorem to specifically separate these two types of forces,

$$W_{nc} + W_c = \Delta KE, \tag{5.3}$$

where we've separated the work done on the object into two parts: the work done by conservative forces and the work done by nonconservative forces.

5.3 Gravitational potential energy

Conservative forces have the nice property that they essentially "store" energy based on their position. When you lift a book, you've done some work on that book and put energy into the book. You can get that energy back by lowering the book, or you can convert that energy to something else (like kinetic energy) by letting go of the book. The book is said to have *potential energy* because it now has the potential to do work on another object.

Let's figure out how much work is done by gravity as a book of mass m falls from y_i to y_f. Remember that the formula for work is

$$W = F\Delta x \cos\theta.$$

The force of gravity is $F_g = -mg$, the displacement is $\Delta x = y_f - y_i$, and the angle between them is $\theta = 0$. So we have,

$$W_g = -mg(y_f - y_i). \tag{5.4}$$

Assuming we have no other conservative forces, we can explicitly put the effect of gravity into the work-energy theorem,

$$\begin{aligned} W_{nc} + W_g &= \Delta KE \\ W_{nc} &= \Delta KE + mg(y_f - y_i) \\ W_{nc} &= \Delta KE + \Delta PE. \end{aligned} \tag{5.5}$$

Note that when you are using the work-energy theorem, it does not matter what you choose as your reference point for measuring the height of an object. It is only changes in height (and therefore changes in gravitational potential energy) that matter and the the actual value of the potential energy at an one point.

In the absence of any nonconservative forces (which will be the case in most of your homework problems), we have

$$\begin{aligned} 0 &= \Delta KE + \Delta PE \\ KE_i + PE_i &= KE_f + PE_f. \end{aligned} \tag{5.6}$$

This is a conservation law — it tells us that the total amount of kinetic energy and potential energy (sometimes called mechanical energy) for a particular system stays the same *all the time.* The amount of kinetic energy and potential energy might change as the object moves, but if you add the energies together, you will always get the same number. A ball sitting on the top of a hill has lots of gravitational potential energy and no kinetic energy. As it starts to roll down, it's potential energy decreases, but it's kinetic energy increases. When it gets to the bottom of the hill, it has no more potential energy, but lots of kinetic energy. So the amount of each type of energy changes, but the total will always be the same.

Example: Platform diver

A diver of mass m drops from a board 10.0 m above the water's surface. Neglect air resistance. (a) Find his speed 5.0 m above the water's surface. (b) Find his speed as he hits the water.

Solution: (a) When solving problems dealing with gravitational potential energy, we need to set a reference point (it doesn't matter where the reference point is, we just need to be consistent for all measurements). Let's choose the bottom of the diving board as $y = 0$. We are told that the diver "drops" from the board, so $v_0 = 0$ which means that his kinetic energy is $KE_i = 0$. The initial position of the diver is at the top of the board where his gravitational potential energy is $PE_i = mgy_i$. The conservation of mechanical energy tells us,

$$
\begin{aligned}
KE_i + PE_i &= KE_f + PE_f \\
0 + mgy_i &= \frac{1}{2}mv_f^2 + mgy_f \\
v_f &= \sqrt{2g(y_i - y_f)} \\
v_f &= \sqrt{2(9.8 \text{ m/s}^2)(10.0 \text{ m} - 5.0 \text{ m})} \\
v_f &= 9.90 \text{ m/s.}
\end{aligned}
$$

(b) We use the same procedure as for part (a), but with a different end point where $y_f = 0$.

$$
\begin{aligned}
KE_i + PE_i &= KE_f + PE_f \\
0 + mgy_i &= \frac{1}{2}mv_f^2 + 0 \\
v_f &= \sqrt{2gy_i} \\
v_f &= \sqrt{2(9.8 \text{ m/s}^2)(10.0 \text{ m})} \\
v_f &= 14.0 \text{ m/s.}
\end{aligned}
$$

Example: Waterslides

Der Stuka is a waterslide at Six Flags in Dallas named for the German dive bombers of World War II. It is 21.9 m high. (a) Determine the speed of a 60.0 kg woman at the bottom of such a slide, assuming no friction is present. (b) If the woman is clocked at 18.0 m/s at the bottom of the slide, find the work done on the woman by friction.

Solution: (a) Let's choose the bottom of the slide as $y = 0$. We can assume that the woman starts from rest, so $v_0 = 0$ which means that her kinetic energy is $KE_i = 0$ at the top of the slide. The initial position of the woman is at the top of the board where her gravitational potential energy is $PE_i = mgy_i$. We are interested in the final position where her height is $y_f = 0$ and so her gravitational potential energy is $PE_f = 0$. The conservation of mechanical energy tells us,

$$\begin{aligned} KE_i + PE_i &= KE_f + PE_f \\ 0 + mgy_i &= \frac{1}{2}mv_f^2 + 0 \\ v_f &= \sqrt{2gy_i} \\ v_f &= \sqrt{2(9.8 \text{ m/s}^2)(21.9 \text{ m})} \\ v_f &= 20.7 \text{ m/s}. \end{aligned}$$

(b) In this case we have to use the full work-energy theorem,

$$\begin{aligned} W_{nc} &= KE_f - KE_i + PE_f - PE_i \\ W_{nc} &= \frac{1}{2}mv_f^2 - 0 + 0 - mgy_i \\ W_{nc} &= \frac{1}{2}(60.0 \text{ kg})(18.0 \text{ m/s})^2 - (60.0 \text{ kg})(9.8 \text{ m/s}^2)(21.9 \text{ m}) \\ W_{nc} &= -3.16 \times 10^3 \text{ J}. \end{aligned}$$

Note that the work done by friction is negative because the woman is losing her energy to friction.

5.4 Spring potential energy

When you compress or stretch a string, you have to apply a force and therefore do some work on the spring. When you move the spring back to its original position, that energy is given back to you. Like gravity, the spring force is a conservative force — any energy you put into the spring when it is stretched or compressed is returned when the spring moves back to its original position. Springs exert a force on an object when they are stretched or compressed and the more you stretch or compress the spring, the larger the force trying to return the spring to its original position. So the force exerted by a spring is proportional to the displacement,

$$F_s = -k\Delta x, \tag{5.7}$$

where k is a proportionality constant called the spring constant (units of newtons per meter). This constant is different for each spring. This equation is often called Hooke's law after Robert Hooke who discovered the relationship. In the case of a spring, we measure the displacement from the equilibrium position of the spring. That is $x = 0$ is the point at which the spring is neither compressed nor stretched. The spring force is sometimes called a *restoring force* because it tries to return the spring to equilibrium.

Calculating the work done by a spring is not as straightforward as calculating the work done by gravity because the size of the spring force changes as the displacement changes (remember the force of gravity is the same no matter the height of the object). The work done by the spring force when an object moves from

x_i to x_f is

$$W_s = -(\frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2). \tag{5.8}$$

We can include this in the work-energy theorem (if there is a spring involved in our system) as part of the work done by conservative forces,

$$W_{nc} = \Delta KE + \Delta PE_g + \Delta PE_s, \tag{5.9}$$

where the potential energy of a spring is $\frac{1}{2}kx^2$.

Example: Block on a spring

A block with mass of 5.00 kg is attached to a horizontal spring with spring constant $k = 4.00 \times 10^2$ N/m. The surface the block rests upon is frictionless. If the block is pulled out to $x_i = 0.05$ m and released, (a) find the speed of the block when it first reaches the equilibrium point. (b) Find the speed when $x = 0.025$ m, and (c) repeat part (a) if friction acts on the block with coefficient $\mu_k = 0.150$.

Solution: (a) In the first part of the problem there is no friction, so there aren't any nonconservative forces and the work-energy theorem can be written as

$$KE_i + PE_{gi} + PE_{si} = KE_f + PE_{gf} + PE_{sf}.$$

We can simplify this a little more by realizing that all the action takes place at the same height, so there are no changes in gravitational potential energy and we can remove that from the equation,

$$KE_i + PE_{si} = KE_f + PE_{sf}.$$

The problem states that the block is "released" at a certain point — this means that the initial velocity is $v_i = 0$. So the initial kinetic energy is also 0. The final position of the object is at the equilibrium point ($x_f = 0$), so the spring potential energy at this point is also 0.

$$\begin{aligned} 0 + \frac{1}{2}kx_i^2 &= \frac{1}{2}mv_f^2 + 0 \\ v_f &= \sqrt{\frac{kx_i^2}{m}} \\ v_f &= \sqrt{\frac{(4.00 \times 10^2 \text{ N/m})(0.05 \text{ m})^2}{5.00 \text{ kg}}} \\ v_f &= 0.45 \text{ m/s}. \end{aligned}$$

(b) This time we're asked for the velocity at a non-equilibrium point. The method we use is still the same, we just have a non-zero final potential energy,

$$\begin{aligned} 0 + \frac{1}{2}kx_i^2 &= \frac{1}{2}mv_f^2 + \frac{1}{2}kx_f^2 \\ v_f &= \sqrt{\frac{k(x_i^2 - x_f^2)}{m}} \\ v_f &= \sqrt{\frac{(4.00 \times 10^2 \text{ N/m})[(0.05 \text{ m})^2 - (0.025 \text{ m})^2]}{5.00 \text{ kg}}} \\ v_f &= 0.39 \text{ m/s}. \end{aligned}$$

(c) Now we add friction, so we will need to consider the energy lost to this nonconservative force. First we need to find the magnitude of the frictional force, so we need a free-body diagram of the block. The block has four forces acting on it: gravity pulling down, the normal force pushing up, the spring force pulling towards equilibrium, and the force of friction pulling away from equilibrium. There is no acceleration in the y direction, so we must have

$$\begin{aligned} \sum F_y &= 0 \\ N - mg &= 0 \\ N &= mg. \end{aligned}$$

We know that the frictional force is related to the normal force,

$$\begin{aligned} f_k &= \mu_k N \\ f_k &= \mu_k mg. \end{aligned}$$

The work done by friction then is

$$\begin{aligned} W_f &= f_k \Delta x \cos\theta \\ W_f &= -\mu_k mg x_i. \end{aligned}$$

The work-energy theorem now has to include the nonconservative work,

$$\begin{aligned} W_{nc} &= \Delta KE + \Delta PE_s \\ -\mu_k mg x_i &= \frac{1}{2}mv_f^2 - \frac{1}{2}kx_i^2 \\ v_f &= \sqrt{\frac{kx_i^2}{m} - 2\mu_k g x_i} \\ v_f &= \sqrt{\frac{(4.00 \times 10^2\ \text{N/m})(0.05\ \text{m})^2}{5.00\ \text{kg}} - 2(0.150)(9.8\ \text{m/s}^2)(0.05\ \text{m})} \\ v_f &= 0.230\ \text{m/s}. \end{aligned}$$

Example: Circus acrobat

A 50.0 kg circus acrobat drops from a height of 2.0 m straight down onto a springboard with a force constant of 8.00×10^3N/m By what maximum distance does she compress the spring?

Solution: There aren't any nonconservative forces in this problem, but spring potential, gravitational potential and kinetic energy all play a role. In this problem, we need to be very clear about how we are measuring distances because there are two displacements that are relevant: her change in height and the compression of the spring. Let's set $y = 0$ to be the point of maximum spring compression and let's call the distance that the spring compresses d. We will call the acrobat's height above the uncompressed springboard h. Now let's use the work-energy theorem,

$$\begin{aligned} KE_i + PE_{gi} + PE_{si} &= KE_f + PE_{gf} + PE_{sf} \\ 0 + mg(h+d) + 0 &= 0 + 0 + \frac{1}{2}kd^2 \\ \frac{1}{2}kd^2 - mgd - mgh &= 0. \end{aligned}$$

We have a quadratic equation and will need to use the quadratic formula,

$$\begin{aligned} d &= \frac{mg \pm \sqrt{m^2g^2 + 2kmgh}}{k} \\ d &= \frac{mg}{k}\left(1 \pm \sqrt{1 + \frac{2kh}{mg}}\right) \\ d &= 0.56 \text{ m } (-0.44). \end{aligned}$$

5.4.1 Power

Power is the rate at which energy is transferred from one object to another. Remember that work is the *amount* of energy transferred from one object to another, so the average power will be the amount of work done over some period of time,

$$\overline{P} = \frac{W}{\Delta t}. \tag{5.10}$$

The unit of power is the Watt (W) or joule/second (J/s). We can write the power in another form by using the definition of work $W = F\Delta x \cos\theta$,

$$\begin{aligned} \overline{P} &= \frac{F\Delta x \cos\theta}{\Delta t} \\ \overline{P} &= F\overline{v}\cos\theta. \end{aligned} \tag{5.11}$$

We can generalize this equation (if we use calculus) to get an equation for the instantaneous power,

$$P = Fv\cos\theta, \tag{5.12}$$

where P and v are the instantaneous power and velocity rather than the average power and velocity.

Example: Shamu

Killer whales are able to accelerate up to 30 mph in a matter of seconds. Neglecting the drag force of water, calculate the average power a killer whale with mass 8.00×10^3 kg would need to generate to reach a speed of 12.0 m/s in 6.00 s.

Solution: To find the power needed by the whale, we need to figure out how much work the whale has to do to reach a speed of 12.0 m/s. We do not know the magnitude of the force needed to generate this acceleration, so we can't use the definition of work. (We also can't find the acceleration using kinematics because we can't assume constant acceleration). The other option is to use the work-energy theorem,

$$\begin{aligned} W_{net} &= \Delta KE \\ W_{net} &= \frac{1}{2}mv_f^2 - 0 \\ W_{net} &= \frac{1}{2}(8.00 \times 10^3 \text{ kg})(12.0 \text{ m/s})^2 \\ W_{net} &= 5.76 \times 10^5 \text{ J.} \end{aligned}$$

We know the elapsed time, so we can use the equation defining average power,

$$\begin{aligned} \overline{P} &= \frac{W}{\Delta t} \\ \overline{P} &= \frac{5.76 \times 10^5 \text{ J}}{6.00 \text{ s}} \\ \overline{P} &= 9.6 \times 10^4 \text{ W.} \end{aligned}$$

You might be wondering why we did not use the equation $\overline{P} = F\overline{v}\cos\theta$. That equation uses the *average* velocity and we are given the final velocity. We can't find the average velocity without assuming constant acceleration, which we can't do in this case.

Chapter 6

Momentum and Collisions

6.1 Momentum and impulse

You probably have an intuitive definition of *momentum*. Objects with a large amount of momentum are hard to stop, i.e. a larger force is required to stop an object with lots of momentum. In physics, of course, we like to have precise definitions for these concepts, so we define the *linear momentum* as

$$\vec{p} = m\vec{v}. \tag{6.1}$$

The linear momentum is proportional to both mass and velocity. That is the more massive an object, the more momentum it has and the faster an object moves the more momentum it has. The unit of momentum is kilogram meter per second (kg · m/s). Notice that momentum is a vector that points in the same direction as an object's velocity. As usual, when dealing with vectors, you will break up momentum into its x and y components,

$$\begin{aligned} p_x &= mv_x \\ p_y &= mv_y. \end{aligned}$$

The momentum is related to the kinetic energy of an object,

$$KE = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m}. \tag{6.2}$$

The concept of momentum is closely tied to the idea of inertia and force. A force is required to change the momentum of an object. We can actually restate Newton's second law in terms of momentum,

$$\vec{F_{net}} = m\vec{a} = m\frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta\vec{p}}{\Delta t}, \tag{6.3}$$

which tells us that the change in momentum over some time is equal to the net force. Two implications arise from this equation. First, if there is no net force, the momentum does not change. Second, to change an object's momentum you need to continuously apply a force over some time period (however small). We call the change in momentum of an object the *impulse*,

$$\vec{I} = \Delta\vec{p} = \vec{F_{net}}\Delta t. \tag{6.4}$$

This is called the impulse-momentum theorem. Most forces vary over time, making it difficult to use the idea of impulse without calculus. We can, however, replace a time-varying force with an average force which is a constant force that delivers the same impulse in the time Δt as the real time-varying force. In this case,

$$\Delta\vec{p} = \vec{F_{av}}\Delta t. \tag{6.5}$$

Example: Car crash

In a crash test, a car of mass 1.50×10^3 kg collides with a wall and rebounds. The initial and final velocities are $v_i = -15.0$ m/s and $v_f = 2.60$ m/s. If the collision lasts for 0.150 s, find (a) the impulse delivered to the car due to the collision and (b) the size and direction of the average force exerted on the car.

Solution: The impulse is the change in momentum,

$$\begin{aligned} I &= p_f - p_i \\ I &= m(v_f - v_i) \\ I &= (1.50 \times 10^3 \text{ kg})(2.60 \text{ m/s} - (-15.0 \text{ m/s})) \\ I &= 2.64 \times 10^4 \text{ kg} \cdot \text{m/s}. \end{aligned}$$

We know that the impulse is related to the average force,

$$\begin{aligned} F_{av} &= \frac{I}{\Delta t} \\ F_{av} &= \frac{2.64 \times 10^4 \text{ kg} \cdot \text{m/s}}{0.150 \text{ s}} \\ F_{av} &= 1.76 \times 10^5 \text{ N}. \end{aligned}$$

6.2 Conservation of momentum

Let's think about what happens when two objects collide. If the two objects are isolated, then we can consider them as a single system. While the two objects will exert forces on each other during the collision, there are no net external forces acting on the system as a whole. If there are no net forces, then the total momentum of the system stays the same throughout the collision process. This is known as the *conservation of momentum.*

Suppose a particle with mass m_1 is travelling with velocity $\vec{v}_{1i}$ towards a particle with mass m_2 which is travelling towards the first particle with velocity $\vec{v}_{2i}$. These two particles will eventually collide. After they collide, the first particle moves away from the second with a velocity $\vec{v}_{1f}$ and the second particle moves away with velocity $\vec{v}_{2f}$. While they are colliding, there will be a contact force between them. The force from particle 2 on particle 1 will change the momentum of particle 1,

$$\vec{F}_{21}\Delta t = m_1\vec{v}_{1f} - m_1\vec{v}_{1i}, \tag{6.6}$$

and similarly, the force from particle 1 on particle 2 will change the momentum of particle 2,

$$\vec{F}_{12}\Delta t = m_2\vec{v}_{2f} - m_2\vec{v}_{2i}. \tag{6.7}$$

By Newton's third law, we know that the contact forces are an action/reaction pair and so they must be equal in magnitude and opposite in direction,

$$\begin{aligned} \vec{F}_{21}\Delta t &= -\vec{F}_{12}\Delta t \\ m_1\vec{v}_{1f} - m_1\vec{v}_{1i} &= -(m_2\vec{v}_{2f} - m_2\vec{v}_{2i}) \\ m_1\vec{v}_{1i} + m_2\vec{v}_{2i} &= m_2\vec{v}_{2f} + m_1\vec{v}_{1f}. \end{aligned} \tag{6.8}$$

This equation tells us that if we add the momenta of the particles before the collision, that will be the same as the sum of all momenta after the collision.

Example: Littering fisherman

A 75 kg fisherman in a 125 kg boat throws a package of mass 15 kg horizontally with a speed of 4.5 m/s. Neglecting water resistance, and assuming the boat is at rest before the package is thrown, find the velocity of the boat after the package is thrown.

Solution: We will treat the fisherman/boat as a single object for the purpose of this problem; the package will be treated as a separate object. Everything is at rest before the package is thrown, so there is no initial momentum. After the package is thrown, the boat/fisherman will have some recoil velocity,

$$\begin{aligned} 0 &= m_b V_b + m_p v_p \\ v_b &= -\frac{m_p v_p}{m_b} \\ v_b &= -\frac{(15\text{ kg})(4.5\text{ m/s})}{200\text{ kg}} \\ v_b &= -0.38\text{ m/s}. \end{aligned}$$

6.2.1 Collisions

While momentum is conserved during a collision, kinetic energy is not necessarily conserved. It is important to stress that *total* energy is always conserved, but certain types of energy, like kinetic energy are not always conserved. During a collision, energy is often lost to friction, sound, heat or deformation of the objects. We can classify collisions into several different types:

Elastic collision: An elastic collision is a collision in which both momentum and kinetic energy are conserved.

Inelastic collision: An inelastic collision is a collision in which momentum is conserved, but kinetic energy is not.

Perfectly inelastic collision: A perfectly inelastic collision is a collision in which momentum is conserved, kinetic energy is not conserved and the two objects stick together and move with the same velocity after the collision.

Example: Truck versus compact

A pickup truck with mass 1.80×10^3 kg is travelling eastbound at 15.0 m/s, while a compact car with mass 9.00×10^2 kg is travelling westbound at -15.0 m/s. The vehicles collide head-on, becoming entangled. (a) Find the speed of the entangled vehicles after the collision. (b) Find the change in the velocity of each vehicle. (c) Find the change in the kinetic energy of the system consisting of both vehicles.

Solution: (a) This problem involves a collision, so we should use conservation of momentum. Before the collision, we know the velocities of both the truck and compact. After the collision, they have the same velocity.

$$\begin{aligned} m_t v_t + m_c v_c &= (m_t + m_c) v \\ v &= \frac{m_t v_t + m_c v_c}{m_t + m_c} \\ v &= \frac{(1.80 \times 10^3 \text{ kg})(15.0 \text{ m/s}) - (9.00 \times 10^2 \text{ kg})(15.0 \text{ m/s})}{1.80 \times 10^3 \text{ kg} + 9.00 \times 10^2 \text{ kg}} \\ v &= 5.0 \text{ m/s}. \end{aligned}$$

(b) The change in velocity of the truck is $\Delta v = v - v_t = 5.0 \text{ m/s} - 15.0 \text{ m/s} = -10 \text{ m/s}$. The change in velocity of the car is $\Delta v = v - v_c = 5.0 \text{ m/s} + 15.0 \text{ m/s} = 20 \text{ m/s}$.
(c) The change in kinetic energy is

$$\begin{aligned} \Delta KE &= KE_f - KE_i \\ \Delta KE &= \frac{1}{2}(m_t + m_c)v^2 - \frac{1}{2} m_t v_t^2 - \frac{1}{2} m_c v_c^2 \\ \Delta KE &= -2.7 \times 10^5 \text{ J}. \end{aligned}$$

Example: Billiard balls

Two billiard balls of identical mass move toward each other. Assume that the collision between them is perfectly elastic. If the initial velocities of the balls are 30.0 cm/s and -20.0 cm/s, what are the velocities of the balls after the collision? Assume friction and rotation are unimportant.

Solution: This problem involves a collision, so we should use conservation of momentum. Before the collision, we know the velocities of both balls. After the collision, both velocities are unknown.

$$\begin{aligned} mv_{1i} + mv_{2i} &= mv_{1f} + mv_{2f} \\ v_{1i} + v_{2i} &= v_{1f} + v_{2f} \\ v_{1i} - v_{1f} &= v_{2f} - v_{2i}. \end{aligned}$$

We only have one equation with two unknowns, so we will need to find another equation. We are told that the collision is perfectly elastic, so we know that kinetic energy is conserved,

$$\begin{aligned} \frac{1}{2}mv_{1i}^2 + \frac{1}{2}mv_{2i}^2 &= \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2 \\ v_{1i}^2 + v_{2i}^2 &= v_{1f}^2 + v_{2f}^2 \\ v_{1i}^2 - v_{1f}^2 &= v_{2f}^2 - v_{2i}^2 \\ (v_{1i} - v_{1f})(v_{1i} + v_{1f}) &= (v_{2f} - v_{2i})(v_{2f} + v_{2i}). \end{aligned}$$

Let's divide the two equations,

$$\begin{aligned} \frac{(v_{1i} - v_{1f})(v_{1i} + v_{1f})}{v_{1i} - v_{1f}} &= \frac{(v_{2f} - v_{2i})(v_{2f} + v_{2i})}{v_{2f} - v_{2i}} \\ v_{1i} + v_{1f} &= v_{2f} + v_{2i}. \end{aligned}$$

This equation is easier to deal with than the one with squared velocities, so let's solve it for v_{1f} and substitute into the conservation of momentum equation

$$v_{1f} = v_{2f} + v_{2i} - v_{1i}$$

$$\begin{aligned} v_{1i} - v_{2f} - v_{2i} + v_{1i} &= v_{2f} - v_{2i} \\ v_{2f} &= v_{1i} \\ v_{2f} &= 30.0 \text{ cm/s} \end{aligned}$$

$$\begin{aligned} v_{1f} &= v_{1i} + v_{2i} - v_{1i} \\ v_{1f} &= v_{2i} \\ v_{1f} &= -20.0 \text{ cm/s}. \end{aligned}$$

The balls have swapped their velocities as if they had passed through each other.

The linear equation derived in the above example

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f}) \tag{6.9}$$

is actually true for any elastic collision, even if the masses are not equal. Instead of using the conservation of kinetic energy, which has quadratic terms and is hard to use, you can use the above linear equation **for an elastic collision**.

6.2.2 Collisions in two dimensions

Most collisions are not head-on collisions with both objects travelling along a single line before and after the collision. Although most collisions occur in three dimensions, we will limit ourselves to two-dimensional collisions. When doing two-dimensional momentum problems, we break up momentum into x and y components, just as we did for Newton's second law. The conservation of momentum tells us that momentum will be conserved in the x direction and in the y direction. That is,we now have two equations

$$\begin{aligned} m_1 v_{1ix} + m_2 v_{2ix} &= m_1 v_{1fx} + m_2 v_{2fx} \\ m_1 v_{1iy} + m_2 v_{2iy} &= m_1 v_{1fy} + m_2 v_{2fy}. \end{aligned} \tag{6.10}$$

There are now three subscripts on the velocity: one telling us which object, one telling us which direction, and one telling us whether it was before or after the collision.

Example 6.8: Collision at an intersection

A car with mass 1.50×10^3 kg travelling east at a speed of 25.0 m/s collides at an intersection with a 2.50×10^3 kg van travelling north at a speed of 20.0 m/s. Find the magnitude and direction of the velocity of the wreckage after the collision, assuming that the vehicles undergo a perfectly inelastic collision and assuming friction between the vehicles and the road can be neglected.

Solution: Before the collision, the car has a velocity only in the x direction and the van has a velocity only in the y direction. After the collision, the velocity of the combined object will have components in both directions,

$$\begin{aligned} m_c v_c + 0 &= (m_c + m_v) v_{fx} \\ 0 + m_v v_v &= (m_c + m_v) v_{fy}. \end{aligned}$$

We can use these equations to find the x and y components of the final velocity

$$\begin{aligned} v_{fx} &= \frac{m_c}{m_c + m_v} v_c = \frac{1.50 \times 10^3 \text{ kg}}{1.50 \times 10^3 \text{ kg} + 2.50 \times 10^3 \text{ kg}} (25.0 \text{ m/s}) = 9.38 \text{ m/s} \\ v_{fy} &= \frac{m_v}{m_c + m_v} v_v = \frac{2.50 \times 10^3 \text{ kg}}{1.50 \times 10^3 \text{ kg} + 2.50 \times 10^3 \text{ kg}} (20.0 \text{ m/s}) = 12.5 \text{ m/s}, \end{aligned}$$

from which we can find the magnitude and direction

$$\begin{aligned} v &= \sqrt{v_{fx}^2 + v_{fy}^2} \\ v &= \sqrt{(9.38 \text{ m/s})^2 + (12.5 \text{ m/s})^2} \\ v &= 15.6 \text{ m/s}. \\ \tan\theta &= \frac{v_{fy}}{v_{fx}} \\ \tan\theta &= \frac{12.5 \text{ m/s}}{9.38 \text{ m/s}} \\ \theta &= 53^\circ. \end{aligned}$$

Example P54: Colliding blocks

Consider a frictionless track as shown in Figure P6.54 of your textbook. A block of mass $m_1 = 5.00$ kg is released from height $h = 5.00$ m. It makes a head-on elastic collision with a block of mass $m_2 = 10.0$ kg that is initially at rest. Calculate the maximum height to which m_1 rises after the collision.

Solution: Let's think about what happens in this problem. The block m_1 starts from some height with no initial velocity and will travel to the bottom of the ramp, gaining speed as it falls. At the bottom of the ramp, it will hit block 2 and transfer some of its speed to block 2. We want to know if block 1 will travel backward after the collision and, if so, how far back up the ramp it travels. Let's consider each step separately.
First, the block falls from a height h. How fast is it moving at the bottom of the ramp? This is an energy conservation problem since all of the block's initial potential energy is converted to kinetic energy (no friction),

$$\begin{aligned} KE_i + PE_i &= KE_f + PE_f \\ 0 + mgh &= \frac{1}{2}m_1 v_f^2 + 0 \\ v_f &= \sqrt{2gh} \\ v_f &= \sqrt{2(9.8\ \text{m/s}^2)(5.00\ \text{m})} \\ v_f &= 9.9\ \text{m/s}. \end{aligned}$$

Now we can look at the collision. Block 1 has initial velocity $v_{1i} = 9.9$ m/s and an unknown final velocity v_{1f}. The second block is initially at rest $v_{2i} = 0$ and has some unknown final velocity v_{2f}. Use conservation of momentum because this is a collision

$$m_1 v_{1i} + 0 = m_1 v_{1f} + m_2 v_{2f}.$$

We have one equation with two unknowns, but we also know that the collision is elastic will give us a second equation. We'll use the linear equation derived earlier instead of the quadratic equation to make the calculation easier,

$$v_{1i} - 0 = -(v_{1f} - v_{2f}).$$

We're actually not too interested in what block 2 does, so let's solve for its final velocity first and substitute into the other equation to find v_{1f}

$$v_{2f} = v_{1i} + v_{1f}$$

$$\begin{aligned} m_1 v_{1i} &= m_1 v_{1f} + m_2(v_{1i} + v_{1f}) \\ v_{1f}(m_1 + m_2) &= v_{1i}(m_1 - m_2) \\ v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \\ v_{1f} &= \frac{5.00\ \text{kg} - 10.0\ \text{kg}}{5.00\ \text{kg} + 10.0\ \text{kg}}(9.9\ \text{m/s}) \\ v_{1f} &= -3.3\ \text{m/s}. \end{aligned}$$

Finally, we do the reverse of what we did in the first part to see how high the block will go

$$\begin{aligned} KE_i + PE_i &= KE_f + PE_f \\ \frac{1}{2}m_1 v_{1f}^2 + 0 &= 0 + mgh \\ h &= \frac{v_{1f}^2}{2g} \\ h &= \frac{(3.3\ \text{m/s})^2}{2(9.8\ \text{m/s}^2)} \\ h &= 0.56\ \text{m}. \end{aligned}$$

Chapter 7

Rotational Motion

All the motion that we have studied to this point is linear motion. All the objects travelled in a straight line (or a series of straight lines). Objects do not always move in a straight line, they often rotate or move in circles. Luckily, many of the concepts you have learned for linear motion have analogues in rotational motion.

7.1 Angular displacement, speed and acceleration

When describing linear motion, the important quantities are displacement Δx, velocity v and acceleration, a. For rotational motion we use the angular displacement $\Delta\theta$, angular velocity ω, and angular acceleration α.

For linear motion, the displacement measured the change in linear position of the object. For rotational motion, we want to measure the net change in angle as the object moves around the circle. You are used to measuring angles in degrees, but a more natural unit for measuring angles is the *radian*. Remember that the circumference of a circle of radius r is $s = 2\pi r$. Rearranging this equation a little gives $s/r = 2\pi$. This quantity is dimensionless, but it tells us that the displacement around *any* circle is 2π. A displacement around half the circle is π; a quarter circle is $\pi/2$. This forms the basis of the unit of radians. Note that we can convert from degrees to radians using the relation $180° = \pi$. Angular quantities in physics must be expressed in radians, so be sure to set your calculators to radian mode when doing rotation problems. We can find the angle travelled by an object rotating at a distance r through an arc length s through

$$\theta = \frac{s}{r}. \tag{7.1}$$

You might wonder why we bother defining a new type of displacement if we can just measure the arc length and use our linear displacement equations. Consider a solid object, called a *rigid body*, like a rotating compact disc where the entire object rotates as a unit. As the object rotates, all points on the CD will have the same angular displacement, but the arc length travelled by points on the disc will vary depending on the radius. Using the notion of angular displacement, we can easily describe the motion of the entire CD, but using linear displacement makes it difficult to treat the CD as a single object.

Let's properly define the angular properties. Suppose an object starts at an angle θ_i and ends at an angle θ_f after some time Δt. The angular displacement is determined by the initial and final angles,

$$\Delta\theta = \theta_f - \theta_i. \tag{7.2}$$

Note that for a rigid body the angular displacement is the same for all points on the object. The unit for angular displacement is the radian (rad).

The average angular velocity of an object is the angular displacement divided by the time,

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t}. \tag{7.3}$$

For a rigid body, again, all points will have the same angular velocity. The units of angular velocity are radian per second (rad/s). We will use the term angular speed when we are not concerned with the direction, but just using the magnitude of the velocity. A positive angular speed denotes counterclockwise rotation and a negative angular speed denotes clockwise rotation. Angular velocity is a vector and the direction is specified by the *right-hand rule*. Take your right hand, curl your fingers in the direction of the motion and your thumb will give the direction of the vector. This rule specifies the rotation axis of the spinning object.

Example: Spinning wheel

A wheel has a radius of 2.0 m. (a) How far does a point on the circumference travel if the wheel is rotated through an angle of 30 rad. (b) If this occurs in 2 s, what is the average angular speed of the wheel?

Solution: (a) We are first asked to find the arc length travelled by a point on the edge of the wheel when the wheel rotates through 30 rad. We use the equation relating the arc length to the angle,

$$\begin{aligned} s &= r\theta \\ s &= (2.0 \text{ m})(30 \text{ rad}) \\ s &= 60 \text{ m}. \end{aligned}$$

(b) Angular speed is the angular displacement divided by time,

$$\begin{aligned} \overline{\omega} &= \frac{\Delta\theta}{\Delta t} \\ \overline{\omega} &= \frac{30 \text{ rad}}{2 \text{ s}} \\ \overline{\omega} &= 15 \text{ rad/s}. \end{aligned}$$

We define the instantaneous angular speed by taking the limit (as we did for velocity),

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}. \tag{7.4}$$

We use this instantaneous angular speed to define the average angular acceleration,

$$\overline{\alpha} = \frac{\Delta\omega}{\Delta t}. \tag{7.5}$$

The units of angular acceleration are radian per second squared (rad/s^2). A rigid body will have the same angular acceleration at all points on the body. The direction of angular acceleration is in the same direction as angular velocity if the object is accelerating, otherwise it is in the opposite direction.

7.1.1 Constant angular acceleration

For linear motion, we developed a number of useful equations when we could assume constant linear acceleration. We can derive similar equations using the angular quantities under the assumption of constant angular acceleration.

Linear motion	**Rotational motion**
$\overline{v} = \frac{v_f + v_i}{2}$	$\overline{\omega} = \frac{\omega_f + \omega_i}{2}$
$v_f = v_i + at$	$\omega_f = \omega_i + \alpha t$
$\Delta x = v_i t + \frac{1}{2}at^2$	$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$
$v_f^2 = v_i^2 + 2a\Delta x$	$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$

Example 7.2: Spinning wheel II

A wheel rotates with a constant angular acceleration of 3.5 rad/s^2. If the angular speed of the wheel is 2.00 rad/s at $t = 0$, (a) through what angle does the wheel rotate between $t = 0$ and $t = 2.00$ s? Give your answer in radians and revolutions. (b) What is the angular speed of the wheel at 2.00 s? (c) What angular displacement (in revolutions) results while the angular speed of part (b) doubles?

Solution: (a) We are told that the angular acceleration is constant, so we can use the equations above. We are given angular acceleration, initial angular speed and a time and we want to find angular displacement,

$$\begin{aligned}\Delta\theta &= \omega_i t + \frac{1}{2}\alpha t^2 \\ \Delta\theta &= (2.00\ \text{rad/s})(2.00\ \text{s}) + \frac{1}{2}(3.5\ \text{rad/s}^2)(2.00\ \text{s})^2 \\ \Delta\theta &= 11.0\ \text{rad}.\end{aligned}$$

To convert this to revolutions, we remember that one revolution is 2π rad, 11.0 rad$/2\pi = 1.75$.

(b) We know want the final angular speed,

$$\begin{aligned}\omega_f &= \omega_i + \alpha t \\ \omega_f &= (2.00\ \text{rad/s}) + (3.5\ \text{rad/s}^2)(2.00\ \text{s}) \\ \omega_f &= 9.00\ \text{rad/s}.\end{aligned}$$

(c) In this case, we have an initial and final angular speed and we want displacement,

$$\begin{aligned}\omega_f^2 &= \omega_i^2 + 2\alpha\Delta\theta \\ \Delta\theta &= \frac{\omega_f^2 - \omega_i^2}{2\alpha} \\ \Delta\theta &= \frac{(18.00\ \text{rad/s})^2 - (9.00\ \text{rad/s})^2}{2(3.5\ \text{rad/s}^2)} \\ \Delta\theta &= 34.7\ \text{rad} = 5.52.\end{aligned}$$

7.1.2 Relations between angular and linear quantities

Remember that we could relate the distance travelled along an arced path to the angular displacement,

$$\Delta s = r\Delta\theta.$$

We can use this relationship to find relationships between other angular and linear quantities. For example, we can divide both sides of this equation by time to get the relationship between angular and linear velocity,

$$\begin{aligned}\frac{\Delta s}{\Delta t} &= r\frac{\Delta\theta}{\Delta t} \\ v_t &= r\omega. \qquad (7.6)\end{aligned}$$

I've used the subscript t here to denote the *tangential* velocity. The instantaneous velocity of the rotating object is always tangent to the circle. That is, if the object were no longer forced to rotate, it would continue in a straight line *tangent to the circle*, as dictated by Newton's first law. So at every point on the path of a rotating object, the velocity is tangent to the path.

We can similarly derive a relationship between the angular and linear accelerations,

$$a_t = r\alpha, \tag{7.7}$$

where again, we use the subscript t to denote the tangential acceleration of the object. In this case, the distinction is very important because, as we shall see shortly, there is a second type of acceleration that is important during rotational motion.

Example 7.3: Germy compact disc

A compact disc rotates from rest up to an angular speed of 31.4 rad/s in a time of 0.892 s. (a) What is the angular acceleration of the disc, assuming the angular acceleration is uniform? (b) Through what angle does the disc turn while coming up to speed? (c) If the radius of the disc is 4.45 cm, find the tangential speed of a microbe riding on the rim of the disc when $t = 0.892$ s. (d) What is the magnitude of the tangential acceleration of the microbe at the given time?

Solution: (a) We are told to assume uniform (constant) angular acceleration, so we can use the equations from the previous section.

$$\begin{aligned} \omega_f &= \omega_i + \alpha t \\ \alpha &= \frac{\omega_f - \omega_i}{t} \\ \alpha &= \frac{31.4 \text{ rad/s}}{0.892 \text{ s}} \\ \alpha &= 35.2 \text{ rad/s}^2. \end{aligned}$$

(b) Now we want to find the angular displacement,

$$\begin{aligned} \omega_f^2 &= \omega_i^2 + 2\alpha\Delta\theta \\ \Delta\theta &= \frac{\omega_f^2 - \omega_i^2}{2\alpha} \\ \Delta\theta &= \frac{(31.4 \text{ rad/s})^2}{2(35.2 \text{ rad/s}^2)} \\ \Delta\theta &= 14.0 \text{ rad}. \end{aligned}$$

(c) Now we use the relationship relating angular velocity and tangential velocity,

$$\begin{aligned} v_t &= r\omega \\ v_t &= (0.0445 \text{ m})(31.4 \text{ rad/s}) \\ v_t &= 1.4 \text{ m/s}. \end{aligned}$$

(d) This time we need to relate the angular acceleration and the tangential acceleration,

$$\begin{aligned} a_t &= r\alpha \\ a_t &= (0.0445 \text{ m})(35.2 \text{ rad/s}^2) \\ a_t &= 1.57 \text{ m/s}^2. \end{aligned}$$

7.2 Centripetal acceleration

We just discussed the relationship between the tangential acceleration and the angular acceleration. The tangential acceleration of an object is determined by changes in how fast an object is spinning. There is another type of acceleration that is present in all rotational motion, even when the rate of rotation is not changing.

Recall that Newton's first law tells us that objects will continue to move *in a straight line* unless acted upon by an outside force. In order for an object to rotate, it must be continuously pulled from the straight line path that it wants to take. The direction of motion of the object is constantly changing, which means that there is some kind of acceleration. This acceleration is called the *centripetal* acceleration. This acceleration is towards the center of rotation and is responsible for changing the direction of motion. The tangential acceleration is responsible for changing the speed of rotation.

The centripetal acceleration is given by

$$a_c = \frac{v^2}{r}, \tag{7.8}$$

where v is the tangential velocity. We can re-write this formula in terms of the angular speed,

$$a_c = \frac{(r\omega)^2}{r} = r\omega^2. \tag{7.9}$$

The total acceleration consists of both the tangential and the centripetal acceleration. Since the two are always perpendicular, the magnitude of the total acceleration is given by

$$a = \sqrt{a_c^2 + a_t^2}. \tag{7.10}$$

Example 7.5: At the racetrack

A race car accelerates uniformly from a speed of 40.0 m/s to a speed of 60.0 m/s in 5.00 s while travelling counterclockwise around a circular track of radius 4.00×10^2 m. When the car reaches a speed of 50.0 m/s, find (a) the magnitude of the car's centripetal acceleration, (b) the angular speed, (c) the magnitude of the tangential acceleration, and (d) the magnitude of the total acceleration.

Solution: (a) The centripetal acceleration can be determined from the tangential speed,

$$\begin{aligned} a_c &= \frac{v^2}{r} \\ a_c &= \frac{(50.0 \text{ m/s})^2}{4.00 \times 10^2 \text{ m}} \\ a_c &= 6.25 \text{ m/s}^2. \end{aligned}$$

(b) The angular speed is related to the tangential speed,

$$\begin{aligned} \omega &= \frac{v}{r} \\ \omega &= \frac{50.0 \text{ m/s}}{4.00 \times 10^2 \text{ m}} \\ \omega &= 0.125 \text{ rad/s}. \end{aligned}$$

(c) The acceleration is uniform, so we can use the constant acceleration equations,

$$\begin{aligned} a_t &= \frac{v_f - v_i}{\Delta t} \\ a_t &= \frac{60.0 \text{ m/s} - 40.0 \text{ m/s}}{5.00 \text{ s}} \\ a_t &= 4.00 \text{ m/s}^2. \end{aligned}$$

(d) The total acceleration is found from the centripetal and tangential accelerations

$$\begin{aligned} a &= \sqrt{a_t^2 + a_c^2} \\ a &= \sqrt{(4.00 \text{ m/s}^2)^2 + (6.25 \text{ m/s}^2)^2} \\ a &= \frac{7.42}{\text{m/s}^2}. \end{aligned}$$

Centripetal force

Since there is an acceleration that is directed towards the center, there must be some force pulling objects towards the center of rotation. This is often called the *centripetal* force, but **this is not actually a new force**. The centripetal force is one of the forces you are already familiar with (friction, normal force, tension, gravity) that happens to be pulling an object towards the center of rotation. In the case of planets orbiting the sun, the centripetal force is gravity. In the case of a yo-yo being swung in a circle, the centripetal force is tension.

When solving rotational motion problems, it is often useful to set up a coordinate system based on the radial and tangential directions of motion. In this case, the net force in the radial direction is the centripetal force and gives rise to the centripetal acceleration,

$$F_c = ma_c = m\frac{v^2}{r}, \tag{7.11}$$

where we have used the formula for centripetal acceleration. If the centripetal force were to disappear, the spinning object would continue travelling in a straight line tangent to the circle.

Centrifugal force

Many people refer to the *centrifugal* force when discussing rotational motion. The centrifugal force is a fictitious force; it does not exist. When we experience rotational motion, we often flee like we are being pushed out from the center of rotation, but this feeling comes about because your body would like to continue in a straight line, tangent to the circle, and you have to exert a centripetal force to continue the rotation.

Example 7.6: Car in a turn

A car travels at a constant speed of 13.4 m/s on a level circular turn of radius 50.0 m. What minimum coefficient of static friction between the tires and the roadway will allow the car to make the circular turn without sliding?

Solution: There are three forces acting on the car: gravity, the normal force and friction. There is a frictional force pointing opposite to the direction of motion of the car, but we are not concerned with that kinetic frictional force. There is another frictional force, static friction in this case, due to the circular path of the car. Due to inertia, the car would like to move tangent to the circle, but friction prevents it from doing so. In this example, friction is the centripetal force.

$$\begin{aligned}
f_s &= m\frac{v^2}{r} \\
\mu_s N &= m\frac{v^2}{r} \\
\mu_s mg &= m\frac{v^2}{r} \\
\mu_s &= \frac{v^2}{gr} \\
\mu_s &= \frac{(13.4\ \mathrm{m/s})^2}{(50.0\ \mathrm{m})(9.8\ \mathrm{m/s^2})} \\
\mu_s &= 0.366.
\end{aligned}$$

Example 7.8: Riding the tracks

A roller coaster car moves around a frictionless circular loop of radius R. (a) What speed must the car have so that it will just make it over the top without any assistance from the track? (b) What speed will the car subsequently have at the bottom of the loop? (c) What will be the normal force on a passenger at the bottom of the loop if the loop has a radius of 10.0 m?

Solution: (a) There are two forces acting on the car: gravity and the normal force, both acting downwards when the car is at the top of the loop. Since we want the car to make it through the loop without the assistance of the track, we set the normal force to 0.

$$\begin{aligned}\sum F_y &= ma_c \\ mg + N &= m\frac{v_t^2}{R} \\ mg &= m\frac{v_t^2}{R} \\ v_t &= \sqrt{gR}.\end{aligned}$$

(b) We can use conservation of energy to find the car's speed at the bottom. At the top, the car has both potential and kinetic energy; at the bottom it only has kinetic energy,

$$\begin{aligned}KE_i + PE_i &= KE_f + PE_f \\ \frac{1}{2}mv_t^2 + mg(2R) &= \frac{1}{2}mv_b^2 + 0 \\ \frac{1}{2}v_b^2 &= \frac{5}{2}gR \\ v_b &= \sqrt{5gR}.\end{aligned}$$

(c) At the bottom of the loop the normal force and the force of gravity are in opposite directions,

$$\begin{aligned}\sum F_y &= ma_c \\ N - mg &= m\frac{v_b^2}{R} \\ N &= m\frac{5gR}{R} + mg \\ N &= 6mg.\end{aligned}$$

7.3 Gravitation

One of the reasons we are interested in circular motion is because we know that astronomical bodies move in (roughly) circular orbits. The centripetal force that causes this rotation is the gravitational force first written by Isaac Newton. His law of universal gravitation states that any two objects will exert an attractive force because of their mass.

If two particles with masses m_1 and m_2 are separated by a distance r, a gravitational force F acts along a line joining them, with magnitude given by

$$F = G\frac{m_1 m_2}{r^2} \tag{7.12}$$

where $G = 6.673 \times 10^{-11}\ \text{kg}^{-1} \cdot \text{m}^3 \cdot \text{s}^{-2}$ is a constant of proportionality called the constant of universal gravitation.

It is important to notice that *both* objects feel an attractive force; gravitational forces form an action-reaction pair.

Example 7.10: Ceres

An astronaut standing on the surface of Ceres, the largest asteroid, drops a rock from a height of 10.0 m. It takes 8.06 s to hit the ground. (a) Calculate the acceleration of gravity on Ceres. (b) Find the mass of Ceres, given that the radius of Ceres is $R_C = 5.1 \times 10^2$ km. (c) Calculate the gravitational acceleration 50.0 km from the surface of Ceres.

Solution: (a) We can use kinematics to find the acceleration,

$$\begin{aligned}
\Delta y &= v_0 t + \frac{1}{2}at^2 \\
a &= \frac{2\Delta y}{t^2} \\
a &= \frac{2(10.0\text{ m})}{(8.06\text{ s})^2} \\
a &= 0.308\text{ m/s}^2.
\end{aligned}$$

(b) This acceleration is caused by the force of gravity pulling on the rock,

$$\begin{aligned}
ma &= G\frac{mM}{R_C^2} \\
M &= \frac{aR_C^2}{G} \\
M &= \frac{(0.308\text{ m/s}^2)(5.1 \times 10^2\text{ km})^2}{6.673 \times 10^{-11}\text{ kg}^{-1} \cdot \text{m}^3 \cdot \text{s}^{-2}} \\
M &= 1.2 \times 10^{21}\text{ kg}.
\end{aligned}$$

(c) From the previous part, we have

$$\begin{aligned}
ma &= G\frac{mM}{R^2} \\
a &= G\frac{M}{R^2} \\
a &= (6.673 \times 10^{-11}\text{ kg}^{-1} \cdot \text{m}^3 \cdot \text{s}^{-2})\frac{1.2 \times 10^{21}\text{ kg}}{(5.1 \times 10^2\text{ km} + 50\text{ km})^2} \\
a &= 0.255\text{ m/s}^2.
\end{aligned}$$

7.3.1 Kepler's Laws

Before Isaac Newton discovered the law of gravity, another astronomer, Johannes Kepler discovered laws that helped him describe the motion of the planets in our solar system. Kepler's three laws state

1. All planets move in elliptical orbits with the Sun at one of the focal points.
2. A line drawn from the Sun to any planet sweeps out equal areas in equal time intervals,
3. The square of the orbital period of any planet is proportional to the cube of the average distance from the planet to the Sun.

It turns out that these laws are a consequence of Newton's gravitational force.

First Law

It turns out that any object moving under the influence of an inverse-square force (force varies as $1/r^2$) will move in an elliptical orbit. Ellipses are curves drawn such that the sum of the distances from any point on the curve to the two foci is constant. For planets in our solar system, the Sun is always at one focus, so a planets distance from the Sun will vary as the planet orbits.

Second Law

Kepler's second law has an interesting consequence. Since the planet is closer to the Sun at some points of its orbit than at other points, in order to sweep out equal area in equal time, it must change its speed as it moves around the Sun. The planet will move more slowly when it is far from the Sun and more quickly when it is close to the Sun.

Third Law

We can derive Kepler's third law from the law of gravity. Suppose that an object (M_p) is moving in a circular orbit around an object (M_s) with a constant velocity (Is this possible given the second law?). We know that the object undergoes centripetal acceleration and that the force causing this acceleration is the gravitational force,

$$M_p a_c = \frac{M_p v^2}{r} = \frac{G M_p M_s}{r^2}.$$

The speed of the object is simply the circumference divided by the time required for one revolution (period)

$$v = \frac{2\pi r}{T}. \tag{7.13}$$

Substituting, we find

$$\begin{aligned} \frac{M_p(4\pi^2 r^2}{rT^2} &= \frac{G M_p M_s}{r^2} \\ T^2 &= \left(\frac{4\pi^2}{G M_s}\right) r^3. \end{aligned} \tag{7.14}$$

Example 7.13: Geosynchronous orbit

From a telecommunications point of view, it's advantageous for satellites to remain at the same location relative to a location on Earth. This can occur only if the satellite's orbital period is the same as the Earth's period of rotation, 24.0 h. (a) At what distance from the center of the Earth can this geosynchronous orbit be found? (b) What's the orbital speed of the satellite?

Solution: (a) We can use Kepler's third law to find the radius of the orbit,

$$\begin{aligned}
T^2 &= \left(\frac{4\pi^2}{GM_s}\right) r^3 \\
r &= \sqrt[3]{\left(\frac{GM_E}{4\pi^2}\right) T^2} \\
r &= \sqrt[3]{\left(\frac{(6.673\times10^{-11}\ \mathrm{kg}^{-1}\cdot\mathrm{m}^3\cdot\mathrm{s}^{-2})(5.98\times10^{24}\ \mathrm{kg}}{4(3.14)^2}\right)(86400\ \mathrm{s})^2} \\
r &= 4.23\times10^7\ \mathrm{m}.
\end{aligned}$$

(b) The speed is simply the circumference divided by the period,

$$\begin{aligned}
v &= \frac{2\pi r}{T} \\
v &= \frac{2(3.14)(4.23\times10^7\ \mathrm{m})}{86400\ \mathrm{s}} \\
v &= 3.08\times10^3\ \mathrm{m/s}.
\end{aligned}$$

7.4 Torque

When we studied forces, we treated all objects as point masses and assumed that it didn't matter where on the object a force was applied. The force would simply move linearly in response to the applied force. This is an extreme simplification of reality, it actually does matter at which point on the object the force is applied. Forces applied near the edges of an extended object will tend to rotate the object rather than move it forward. If you lay your textbook on the table and push it with a force applied near the center of the book, it will slide forward. If you push the book with a force applied near the edge, it will rotate rather than move forward.

Remember that forces cause an acceleration. Similarly, we define something called a *torque* which causes an angular acceleration. Forces and torques are not completely independent — forces cause torques, but torques also depend on where the force is applied and on the angle at which the force is applied.

Let $\vec{F}$ be a force acting on an object, and let $\vec{r}$ be a position vector from the point of rotation to the point of application of the force. The magnitude of the torque $\vec{\tau}$ exerted by the force $\vec{F}$ is

$$\tau = rF\sin\theta \tag{7.15}$$

where r is the length of the position vector, F is the magnitude of the force and θ is the angle between $\vec{r}$ and $\vec{F}$. The unit of torque is newton-meter.

Torque is a vector with the direction given by the right hand rule. Point your fingers in the direction of $\vec{r}$ and curl them toward the direction of $\vec{F}$. Your thumb will point in the direction of the torque. This will be perpendicular to the plane that contains both $\vec{r}$ and $\vec{F}$. When your fingers point in the direction of the torque, your fingers will curl in the direction of rotation that the torque will cause.

Example 8.2: Swinging door

(a) A man applies a force of $F = 3.00 \times 10^2$ N at an angle of 60.0° to the door 2.00 m from the hinges. Find the torque on the door, choosing the position of the hinges as the axis of rotation. (b) Suppose a wedge is placed at 1.50 m from the hinges on the other side of the door. What minimum force must the wedge exert so that the force applied in part (a) won't open the door?

Solution: (a) We use the above equation to calculate the torque on the door

$$\begin{aligned} \tau &= Fr\sin\theta \\ \tau &= (F = 3.00 \times 10^2 \text{ N})(2.00 \text{ m})\sin(60.0^\circ) \\ \tau &= 520 \text{ N}\cdot\text{m}. \end{aligned}$$

The direction of torque is out of the board/page (towards top of door). (b) We don't want the door to rotate, so there must be no net torque on the door. We need to identify all the torques/forces acting on the door. There are three forces acting on the door: the applied force, the force of the wedge and the force of the hinges. Although the hinges apply a force to the door, they do not exert a torque because they act at the point of rotation.

$$\begin{aligned} \tau_{\text{hinges}} + \tau_{\text{wedge}} + \tau &= 0 \\ 0 + F_{\text{wedge}} r_{\text{wedge}} \sin(-90^\circ) + \tau &= 0 \\ F_{\text{wedge}} &= \frac{\tau}{r_{\text{wedge}}\sin(-90^\circ)} \\ F_{\text{wedge}} &= \frac{520 \text{ N}\cdot\text{m}}{(1.50 \text{ m})\sin(-90^\circ)} \\ F_{\text{wedge}} &= 347 \text{ N}. \end{aligned}$$

7.4.1 Equilibrium

An object that is in equilibrium must satisfy two conditions,

1. The net force must be zero, i.e. *no linear acceleration* ($\sum F = 0$).
2. The net torque must be zero, i.e. *no angular acceleration* ($\sum \tau = 0$).

This does not mean that the object is not moving or rotating, it can be moving at a constant velocity or rotating at a constant angular speed. If an object is in equilibrium, we can choose the axis of rotation so we should choose one which makes the calculation convenient. An axis where at least one torque is zero makes calculations easier.

Example 8.3: Balancing act

A woman of mass $m = 55.0$ kg sits of the left side of a seesaw — a plank of length $L = 4.00$ m, pivoted in the middle. (a) First compute the torques on the seesaw about an axis that passes through the pivot point. Where should a man of mass $M = 75.0$ kg sit if the system is to be balanced? (b) Find the normal force exerted by the pivot if the plank has a mass of $m_p = 12.0$ kg. (c) Repeat part (a), but this time compute the torques about an axis through the left end of the plank.

Solution: (a) We need to identify all the forces acting on the plank and their point of application. The woman and the man will push down on the plank with a force equal to their respective weights. We know the woman's distance from the center of the plank, but the man's distance is unknown. Gravity also acts on the plank itself, and will pull down on the center of the plank as if all the plank's mass was concentrated at that one point. Finally, there is a normal force that pushes upwards from the pivot point. We want the seesaw to be in equilibrium, so the sum of all torques about some axis of rotation (the pivot point in this case) must be zero,

$$\begin{aligned}
\sum \tau &= 0 \\
\tau_N + \tau_g + \tau_w + \tau_m &= 0 \\
0 + 0 + mg\frac{L}{2} - Mgx &= 0 \\
x &= \frac{mL}{2M} \\
x &= \frac{(55.0\text{ kg})(4.00\text{ m})}{2(75.0\text{ kg})} \\
x &= 1.47\text{ m.}
\end{aligned}$$

(b) We now want to find the normal force. We can use the first condition of equilibrium to do this,

$$\begin{aligned}
\sum F &= 0 \\
-Mg - mg - m_p g + N &= 0 \\
N &= (M + m + m_p)g \\
N &= (75.0\text{ kg} + 55.0\text{ kg} + 12.0\text{ kg})(9.8\text{ m/s}^2) \\
N &= 1.39 \times 10^3\text{ N.}
\end{aligned}$$

(c) We will now use the left end of the plank (where the woman sits) as the axis of rotation,

$$\begin{aligned}
\sum \tau &= 0 \\
\tau_N + \tau_g + \tau_w + \tau_m &= 0 \\
-N\frac{L}{2} + m_p g\frac{L}{2} + 0 + Mg(x + \frac{L}{2}) &= 0 \\
x &= \frac{(N - m_p g - Mg)\frac{L}{2}}{Mg} \\
x &= \frac{(Mg + mg + m_p g - m_p g - Mg)\frac{L}{2}}{Mg} \\
x &= \frac{mL}{2M} \\
x &= 1.47\text{ m.}
\end{aligned}$$

7.4.2 Center of gravity

As we saw in the previous problem, we treat gravity as if it acts on a single point of an extended body. This point is called the *center of gravity*. Suppose we have an object with some arbitrary shape. We can treat this object as if it is divided into very small pieces of weights $m_1 g$, $m_2 g$ ... at locations (x_1,y_1), (x_2,y_2).... Each piece contributes some torque about the axis of rotation due to its weight. For example, $\tau_1 = m_1 g x_1$ and so forth.

The center of gravity is the point where we apply a single force of magnitude $F = \sum_i m_i g$ which has the same effect on the rotation of the object as all the individual little pieces.

$$\begin{aligned} F x_{cg} &= \sum_i \tau_i \\ (\sum_i m_i g) x_{cg} &= \sum_i m_i g x_i \\ x_{cg} &= \frac{\sum_i m_i x_i}{\sum_i m_i}. \end{aligned} \tag{7.16}$$

This gives us the x coordinate of the center of gravity. We can find the y and z coordinates in a similar fashion,

$$\begin{aligned} y_{cg} &= \frac{\sum_i m_i y_i}{\sum_i m_i} \\ z_{cg} &= \frac{\sum_i m_i z_i}{\sum_i m_i}. \end{aligned} \tag{7.17}$$

If an object is symmetric, the center of gravity will lie on the axis of symmetry, so it is sometimes possible to guess where the center of gravity is for such objects (like we did for the plank in the example problem).

Example 8.4: Center of gravity

Three objects are located on the x-axis as follows: a 5.00 kg mass sits at $x = -0.500$ m, a 2.00 kg mass sits at the origin, and a 4.00 kg mass sits at $x = 1.00$ m. Find the center of gravity. (b) How does the answer change if the object on the left is displaced upward by 1.00 m and the object on the right is displaced downward by 0.50 m?

Solution: (a) We simply apply the formula for center of gravity that we just derived,

$$\begin{aligned} x_{cg} &= \frac{\sum_i m_i x_i}{\sum_i m_i} \\ x_{cg} &= \frac{(5.00\text{ kg})(-0.500\text{ m}) + (2.00\text{ kg})(0) + (4.00\text{ kg})(1.00\text{ m})}{5.00\text{ kg} + 2.00\text{ kg} + 4.00\text{ kg}} \\ x_{cg} &= 0.136\text{ m}. \end{aligned}$$

(b) The x coordinate of the center of gravity will not change since the masses have not been moved along the x-axis. We will, however, have to consider the y axis now,

$$\begin{aligned} y_{cg} &= \frac{\sum_i m_i y_i}{\sum_i m_i} \\ y_{cg} &= \frac{(5.00\text{ kg})(1.00\text{ m}) + (2.00\text{ kg})(0) + (4.00\text{ kg})(-0.500\text{ m})}{5.00\text{ kg} + 2.00\text{ kg} + 4.00\text{ kg}} \\ y_{cg} &= 0.273\text{ m}. \end{aligned}$$

7.5 Torque and angular acceleration

When an object is subjected to a torque, it undergoes an angular acceleration. We can derive a law similar to Newton's second law for the effect of a torque. Suppose we have an object of mass m connected to a very light rod of length r. The rod is pivoted about the end opposite the mass and its movement is confined to a horizontal frictionless table. Suppose a tangential force F_t acts on the mass. This will cause a tangential acceleration,

$$F_t = ma_t.$$

Multiplying both sides of the equation by r,

$$F_t r = mra_t,$$

and substituting for the $a_t = r\alpha$ for the tangential acceleration gives

$$F_t r = mr^2\alpha. \tag{7.18}$$

The left side is simply the torque,

$$\tau = (mr^2)\alpha. \tag{7.19}$$

This tells us that the torque is proportional to the angular acceleration. The constant of proportionality is mr^2 and is called the *moment of inertia* of the object. Moment of inertia has units of $\mathrm{kg}\cdot\mathrm{m}^2$ and is denoted by I. So we can write,

$$\tau = I\alpha. \tag{7.20}$$

This is the rotational analog of Newton's second law.

7.5.1 Moment of inertia

The formula for moment of inertia that we just derived is true for a single point mass only. For extended objects, the moment of inertia will be different. Consider a solid object rotating about its axis. We can break this object up into many little pieces like we did to find the center of gravity. The net torque on the object will be the sum of the torques caused by all the small pieces,

$$\sum_i \tau_i = (\sum_i m_i r_i^2)\alpha. \tag{7.21}$$

The moment of inertia of the whole object then is

$$I = \sum_i m_i r_i^2. \tag{7.22}$$

We can find the moment of inertia of any object or any collection of objects by adding the moments of inertia of its constituents. Notice that the moment of inertia depends not just on the mass of an object, but on how the mass is distributed within the object. Importantly, it matters how the mass is distributed relative to the axis of rotation.

Table 7.1: Moments of inertia for various rigid objects of uniform composition

Object	Axis of Rotation	Moment of inertia
Hoop or cylindrical shell	center	$I = MR^2$
Solid sphere	center	$I = \frac{2}{5}MR^2$
Solid cylinder or disk	center	$I = \frac{1}{2}MR^2$
Thin spherical shell	center	$I = \frac{2}{3}MR^2$
Long thin rod	center	$I = \frac{1}{12}ML^2$
Long thin rod	end	$I = \frac{1}{3}ML^2$

Example 8.9: Baton twirler

In an effort to be the star of the halftime show, a majorette twirls an unusual baton made up of four spheres fastened to the ends of very light rods. Each rod is 1.0 m long. Two of the spheres have a mass of 0.20 kg and the other two spheres have a mass of 0.30 kg. Spheres of equal masses are placed across from each other. (a) Find the moment of inertia of the baton through the point where the rods cross. (b) The majorette tries spinning her strange baton about the rod holding the 0.2 kg spheres. Calculate the moment of inertia of the baton about this axis.

Solution: (a) When the baton is spinning around the point where the rods cross, all four spheres contribute to the moment of inertia. We can treat the spheres as point masses since their radius is small compared to the length of the rods.

$$\begin{aligned}
I &= \sum_i m_i r_i^2 \\
I &= 2m_1 r^2 + 2m_2 r^2 \\
I &= 2(0.20 \text{ kg})(0.5 \text{ m})^2 + 2(0.30 \text{ kg})(0.5 \text{ m})^2 \\
I &= 0.25 \text{ kg} \cdot \text{m}^2.
\end{aligned}$$

(b) In this case only the 0.3 kg spheres contribute to the moment of inertia because the 0.2 kg spheres lie along the axis of rotation (so $r = 0$).

$$\begin{aligned}
I &= \sum_i m_i r_i^2 \\
I &= 2m_2 r^2 \\
I &= (0.30 \text{ kg})(0.5 \text{ m})^2 \\
I &= 0.15 \text{ kg} \cdot \text{m}^2.
\end{aligned}$$

The moment of inertia for solid extended objects can be calculated using calculus. The moment of inertia for some common objects is given in Table 7.1. Note that the assumption is that the mass is distributed uniformly throughout these objects.

Parallel axis theorem

A useful property of the moment of inertia is that it is fairly easy to calculate the moment of inertia about an axis parallel to the axis through the center of gravity of the object. This result is called the parallel axis theorem and is as follows,

$$I_z = I_{cm} + Md^2, \tag{7.23}$$

where I_{cm} is the moment of inertia of the object rotating about the center of mass, M is the mass of the object and d is the distance between the two parallel axes.

Example 8.11: Falling bucket

A solid uniform frictionless cylindrical reel of mass $M = 3.00$ kg and radius $R = 0.400$ m is used to draw water from a well. A bucket of mass $m = 2.00$ kg is attached to a cord that is wrapped around the cylinder. (a) Find the tension T in the cord and acceleration a of the bucket. (b) If the bucket starts from rest at the top of the well and falls for 3.00 s before hitting the water, how far does it fall?

Solution: (a) We will need free body diagrams for both the wheel and the bucket. The bucket has two forces acting on it: tension pulling up and gravity pulling down. Note that we don't care where these forces act on the bucket because this object is not rotating. The cylinder has three forces acting on it: gravity acting at the center and pulling down, a normal force (from the bar holding the cylinder) also acting at the center and pushing up, and tension acting at a distance R and pulling down. We know that the bucket is accelerating and the cylinder has an angular acceleration. We can use Newton's second law on the bucket,

$$\begin{aligned} \sum F &= ma \\ mg - T &= ma. \end{aligned}$$

We can use the rotational analog of Newton's second law on the cylinder. In this case, we don't get to choose the point of rotation because the object is rotating about a specific axis,

$$\begin{aligned} \sum \tau &= I\alpha \\ TR &= \frac{1}{2}MR^2\alpha \\ T &= \frac{1}{2}MR\alpha. \end{aligned}$$

Gravity and the normal force don't contribute to the torque because they act at the axis of rotation. We now have two equations and three unknowns (a, α and T). We will need one more equation to solve this problem. Remember that the tangential acceleration is related to the angular acceleration of a rotating object. In this case, the rope is causing the tangential acceleration of the cylinder and we know that the acceleration of the rope is the same as that of the bucket,

$$a = R\alpha.$$

Using this relationship, we find

$$T = \frac{1}{2}Ma,$$

which we can use to substitute into the first equation,

$$\begin{aligned} mg - \frac{1}{2}Ma &= ma \\ a(m + \frac{1}{2}M) &= mg \\ a &= \frac{mg}{(m + \frac{1}{2}M)} \\ a &= \frac{(2.00 \text{ kg})(9.8 \text{ m/s}^2)}{2.00 \text{ kg} + \frac{1}{2}(3.00 \text{ kg})} \\ a &= 5.60 \text{ m/s}^2. \end{aligned}$$

Now we can also find the tension,

$$T = \frac{1}{2}Ma$$

$$\begin{aligned} T &= \frac{1}{2}(3.00 \text{ kg})(5.60 \text{ m/s}^2) \\ T &= 8.4 \text{ N}. \end{aligned}$$

(b) This is a kinematics problem,

$$\begin{aligned}\Delta y &= v_0 t + \frac{1}{2}at^2 \\ \Delta y &= \frac{1}{2}(5.60\ \text{m/s}^2)(3.0\ \text{s})^2 \\ \Delta y &= 25.2\ \text{m}.\end{aligned}$$

7.6 Rotational kinetic energy

Recall that an object moving through space has kinetic energy. Similarly, a rotating object will have rotational kinetic energy. Consider the mass connected to a light rod rotating on a horizontal frictionless table. The kinetic energy of the mass is

$$KE = \frac{1}{2}mv^2.$$

We know that the velocity is related to the angular speed,

$$KE = \frac{1}{2}m(r\omega)^2 = \frac{1}{2}(mr^2)\omega^2 = \frac{1}{2}I\omega^2. \tag{7.24}$$

Notice that again the equations for translational (linear) kinetic energy and rotational kinetic energy are quite similar with moment of inertia replacing mass and angular speed replacing linear velocity.

In the case of the rotating mass on a rod, either expression can be used to describe it's energy because it only undergoes rotational motion. There are cases, however when *both* expressions are used such as when balls or wheels are rolling. In this case, there is rotation about the center of mass while the center of mass itself is moving through space. The translational kinetic energy refers to the energy of the center of mass' motion while the rotational kinetic energy refers to the energy of the rotation.

This new type of energy needs to be included in the work-energy theorem,

$$W_{nc} = \Delta KE_t + \Delta KE_r + \Delta PE. \tag{7.25}$$

Example 8.12: Ball on an incline

A ball of mass M and radius R starts from rest at a height of 2.00 m and rolls down a 30° slope. What is the linear speed of the ball when it leaves the incline? Assume that the ball rolls without slipping.

Solution: We can use energy to solve this problem. Let's consider the energy at the top and bottom of the ramp, remembering that the ball rolls (rotates) down the ramp,

$$\begin{aligned} KE_{ti} + KE_{ri} + PE_g &= KE_{tf} + KE_{rf} + PE \\ 0 + 0 + Mgh &= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 + 0. \end{aligned}$$

Remember that there is a relationship between the translational velocity and the angular speed. Note that this relationship will only hold if the object "rolls without slipping." If the ball slips then the center of mass moves while the object is not rotating and the relationship does not hold.

$$\begin{aligned} Mgh &= \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v}{R}\right)^2 \\ gh &= \frac{1}{2}v^2 + \frac{1}{5}v^2 \\ v &= \sqrt{\frac{10}{7}gh} \\ v &= \sqrt{\frac{10}{7}(9.8\text{ m/s}^2)(2.00\text{ m})} \\ v &= 5.29\text{ m/s}. \end{aligned}$$

The velocity is smaller than the velocity of a block sliding down the incline because some of the gravitational potential energy goes into rotational kinetic energy.

7.7 Angular momentum

When we apply a torque to an object, we change its angular acceleration — and we have an equation relating the two. Just as we re-wrote Newton's second law in terms of momentum, we can re-write the rotational equivalent in terms of *angular momentum*.

$$\sum \tau = I\alpha = I\frac{\Delta\omega}{\Delta t} = \frac{I\Delta\omega}{\Delta t} = \frac{\Delta L}{\Delta t}, \tag{7.26}$$

where we have defined $L = I\omega$ as the angular momentum of an object. If there is no net torque, then the total angular momentum of a system does not change, $L_i = L_f$. The law of conservation of angular momentum explains why figure skaters spin faster when they bring their arms closer to their bodies. As their arms move in, their moment of inertia decreases, so their angular speed must increase to compensate. Note that angular momentum is a vector with the direction determined by the direction of angular velocity. Changes in direction of angular momentum (changes in the direction of the axis of rotation) are also subject to conservation of momentum. If the axis of rotation changes, there must be a change in the angular momentum of some other part of the system to compensate.

Example 8.14: Merry-go-round

A merry-go-round modeled as a disk of mass $M = 1.00 \times 10^2$ kg and radius $R = 2.00$ m is rotating in a horizontal plane about a frictionless vertical axle. (a) After a student with mass $m = 60.0$ kg jumps on the rim of the merry-go-round, the system's angular speed decreases to 2.00 rad/s. If the student walks slowly from the edge toward the center, find the angular speed of the system when she reaches a point 5.00 m from the center. (b) Find the change in the system's rotational kinetic energy caused by her movement to $r = 0.500$ m. (c) Find the work done on the student as she walks to $r = 0.500$ m.

Solution: (a) There are two parts to the moment of inertia of the system, the moment of inertia of the disk and the moment of inertia of the person. It is the moment of inertia of the student that changes as she walks towards the center — the moment of inertia of the disk remains the same,

$$
\begin{aligned}
L_i &= L_f \\
(I_d + I_{pi})\omega_i &= (I_d + I_{pf})\omega_f \\
\omega_f &= \frac{(I_d + I_{pi})\omega_i}{I_d + I_{pf}} \\
\omega_f &= \frac{(\frac{1}{2}MR^2 + mR^2)\omega_i}{\frac{1}{2}MR^2 + mr^2} \\
\omega_f &= \frac{(\frac{1}{2}(1.00 \times 10^2 \text{ kg})(2.00 \text{ m})^2 + (60.0 \text{ kg})(2.00 \text{ m})^2)(2.00 \text{ rad/s})}{\frac{1}{2}(1.00 \times 10^2 \text{ kg})(0.500 \text{ m})^2} \\
\omega_f &= 4.09 \text{ rad/s}.
\end{aligned}
$$

(b) The change in rotational kinetic energy is

$$
\begin{aligned}
\Delta KE_r &= \frac{1}{2}(I_d + I_{pf})\omega_f^2 - \frac{1}{2}(I_d + I_{pi})\omega_i^2 \\
\Delta KE_r &= \frac{1}{2}(\frac{1}{2}(1.00 \times 10^2 \text{ kg})(0.500 \text{ m})^2 + (60.0 \text{ kg})(2.00 \text{ m})^2)(4.09 \text{ rad/s})^2 \\
&- \frac{1}{2}((\frac{1}{2}(1.00 \times 10^2 \text{ kg})(2.00 \text{ m})^2 + (60.0 \text{ kg})(2.00 \text{ m})^2)(2.00 \text{ rad/s}))^2 \\
\Delta KE_r &= 920 \text{ J}.
\end{aligned}
$$

(c) When calculating the work done by the student we need to use the change in kinetic energy of the student only,

$$
\begin{aligned}
\Delta KE_r &= \frac{1}{2}I_{pf}\omega_f^2 - \frac{1}{2}I_{pi}\omega_i^2 \\
\Delta KE_r &= \frac{1}{2}(60.0 \text{ kg})(2.00 \text{ m})^2(4.09 \text{ rad/s})^2 - \frac{1}{2}(60.0 \text{ kg})(2.00 \text{ m})^2)(2.00 \text{ rad/s}))^2 \\
\Delta KE_r &= -355 \text{ J}.
\end{aligned}
$$

Chapter 8

Vibrations and Waves

We have now studied linear motion and circular motion. There is one more very important type of motion that arises in many aspects of physics. Periodic motion, such as waves or vibrations underlies sound and light and many other physical phenomena.

8.1 Return of springs

One simple type of periodic motion is an object attached to a spring. Remember that the force of a spring is given by Hooke's law,

$$F = -kx. \tag{8.1}$$

This force is sometimes called a *restoring force* because it likes to pull the object back to the equilibrium position. The negative sign ensures that the force is pulling opposite to the direction of displacement. Suppose we pull the object so that the spring is stretched and let go. The spring force will cause an acceleration back towards the equilibrium position. The object will pick up speed as it moves back towards equilibrium and will overshoot the equilibrium position. Once it passes the equilibrium position the object starts to compress the spring and the force changes direction. The force now decelerates the object, eventually causing the object to stop. When the object stops, the spring is compressed and the force still points towards the equilibrium. So the object will accelerate towards the center again. In this way and object will move back and forth endlessly.

This is an example of *simple harmonic motion.* Simple harmonic motion occurs when the net force along the direction of motion obeys Hooke's Law. Not all periodic motion is simple harmonic motion. Two people tossing a ball back and forth is not simple harmonic motion even though it is periodic. The force causing the motion of the ball is not of the form of Hooke's Law, so it cannot be simple harmonic motion.

The acceleration of an object undergoing simple harmonic motion can be found using Newton's second law,

$$\begin{aligned} F &= ma \\ -kx &= ma \\ a &= -\frac{k}{m}x. \end{aligned} \tag{8.2}$$

8.1.1 Energy of simple harmonic motion

Let's consider the energy of an object attached to a spring. Suppose that we pull the object and stretch the spring then release it. Just before the object is released, the spring is at it's maximum stretch. This is called the *amplitude.* The energy at this point is

$$E = \frac{1}{2}kA^2, \tag{8.3}$$

where A is the amplitude. Now we release the spring and as it moves it picks up speed. The object now has both potential energy and kinetic energy, so

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2. \tag{8.4}$$

We can use this to find the velocity at any position,

$$\begin{aligned} \frac{1}{2}kA^2 &= \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \\ v &= \pm\sqrt{\frac{k}{m}(A^2 - x^2)}. \end{aligned} \tag{8.5}$$

The $\pm$ appears because of the square root. The usual convention is that if the object moves to the right, the velocity is positive; if it moves to the left, it is negative.

8.1.2 Connecting simple harmonic motion and circular motion

When the object on the spring moves back and forth, it's similar to an object moving with constant angular velocity around a circle. The object moving around the circle will come back to its original position at regular time intervals, just like the mass on a spring. Remember that the period of a rotating object is

$$T = \frac{2\pi r}{v}. \tag{8.6}$$

For the rotating object, r is the size of the spatial displacement and corresponds to the amplitude of the mass on a spring,

$$T = \frac{2\pi A}{v}.$$

The velocity of the mass after it has travelled a distance A can be found from Eq. (8.5) by setting $x = 0$,

$$v = A\sqrt{\frac{k}{m}}.$$

Now we can put this into the equation for the period,

$$T = 2\pi\sqrt{\frac{m}{k}}. \tag{8.7}$$

This represents the time it takes for a mass on a spring to return to its starting position. A larger mass gives a longer period, while a larger spring constant (stiffer spring) gives a shorter period.

The *frequency* is the inverse of the period

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{8.8}$$

The units of frequency are cycles per second or Hz. This is related the angular frequency (which is in radians per second),

$$\omega = 2\pi f = \sqrt{\frac{k}{m}}. \tag{8.9}$$

Example 13.5: Shock absorbers

A 1.3×10^3 kg car is constructed on a frame supported by four springs. Each spring has a spring constant of 2.00×10^4 N/m. If two people riding in the car have a combined mass of 1.6×10^2 kg, find the frequency of vibration of the car when it is driven over a pothole. Find also the period and the angular frequency. Assume the weight is evenly distributed.

Solution: First we need to find the total mass,

$$m_t = m_c + m_p = 1.3 \times 10^3 \text{ kg} + 1.6 \times 10^2 \text{ kg} = 1.46 \times 10^3 \text{ kg}$$

Each spring will hold up one quarter of the total mass. The frequency is

$$\begin{aligned} f &= \frac{1}{2\pi}\sqrt{\frac{k}{m}} \\ f &= \frac{1}{2\pi}\sqrt{\frac{2.00 \times 10^4 \text{ N/m}}{365 \text{ kg}}} \\ f &= 1.18 \text{ Hz.} \end{aligned}$$

The period is the inverse of frequency

$$T = \frac{1}{f} = \frac{1}{1.18 \text{ Hz}} = 0.847 \text{ s},$$

and the angular frequency is

$$\omega = 2\pi f = 2\pi(1.18 \text{ Hz}) = 7.41 \text{ rad/s}.$$

8.2 Position, velocity and acceleration

Suppose a mass is moving on a circle with constant angular velocity. If we look at it's x position as it moves around the circle, we see that it oscillates somewhat like a mass on a spring. The x position of the mass is given by

$$x = A\cos\theta. \tag{8.10}$$

We know that the mass is moving with constant angular speed so

$$x = A\cos(\omega t) = A\cos(2\pi f t). \tag{8.11}$$

This equation describes the position of an object undergoing simple harmonic motion as a function of time. We can substitute this into Eq. (8.5)

$$\begin{aligned} v &= \pm\sqrt{\frac{k}{m}(A^2 - x^2)} \\ v &= \pm\sqrt{\frac{k}{m}(A^2 - (A\cos(2\pi f t))^2} \\ v &= A\omega\sqrt{1 - \cos^2(2\pi f t)} \\ v &= A\omega \sin(2\pi f t). \end{aligned} \tag{8.12}$$

The velocity also oscillates, but it is 90° out of phase with the displacement. When the displacement is a maximum or minimum, velocity is zero and vice versa. The maximum value (amplitude) of velocity is $A\omega$

(when $\sin(\pi f t) = 1$). We can also derive an expression for the acceleration

$$\begin{aligned} a &= -\frac{k}{m}x \\ a &= -A\omega^2 \cos\omega(2\pi f t). \end{aligned} \tag{8.13}$$

The acceleration is also sinusoidal and 180° out of phase with the displacement. When the displacement is a maximum, acceleration is a minimum and vice versa. The maximum acceleration (amplitude) is $A\omega^2$.

Example 13.6: Vibrating system

(a) Find the amplitude, frequency, and period of motion for an object vibrating at the end of a horizontal spring if the equation for its position as a function of time is

$$x = (0.250 \text{ m}) \cos\left(\frac{\pi}{8.00} t\right).$$

(b) Find the maximum magnitude of the velocity and acceleration. (c) What are the position, velocity, and acceleration of the object after 1.00 s has elapsed?

Solution: (a) Compare the given function to the standard function for simple harmonic motion

$$x = A \cos(2\pi f t)$$

and we can just read off the amplitude,

$$A = 0.250 \text{ m},$$

and the frequency

$$\begin{aligned} 2\pi f &= \frac{\pi}{8.00} \\ f &= \frac{1}{16} = 0.0625 \text{ Hz}. \end{aligned}$$

The period is the inverse of frequency

$$T = \frac{1}{f} = \frac{1}{0.0625 \text{ Hz}} = 16 \text{ s}.$$

(b) The maximum velocity is

$$\begin{aligned} v_{max} &= A\omega \\ v_{max} &= (0.250 \text{ m})(2\pi)(0.0625 \text{ Hz}) \\ v_{max} &= 0.098 \text{ m/s}, \end{aligned}$$

and the maximum acceleration is

$$\begin{aligned} a_{max} &= A\omega^2 \\ a_{max} &= (0.250 \text{ m})(2\pi)^2(0.0625 \text{ Hz})^2 \\ a_{max} &= 0.039 \text{ m/s}^2. \end{aligned}$$

(c) We simply substitute into the given equation

$$x = (0.250 \text{ m}) \cos\left(\frac{\pi}{8.00}\right) = 0.231 \text{ m},$$

and for the velocity

$$v = -(0.098 \text{ m}) \cos\left(\frac{\pi}{8.00}\right) = -0.038 \text{ m/s},$$

and for the acceleration

$$a = -(0.039 \text{ m}) \cos\left(\frac{\pi}{8.00}\right) = -0.036 \text{ m/s}^2.$$

8.3 Motion of a pendulum

Another type of periodic motion that you may have observed is that of a pendulum swinging back and forth. To determine if it is simple harmonic motion, we need to figure out whether there is a Hooke's law type force causing the pendulum to move. There are two forces acting on the pendulum: the force of gravity pulls down and tension pulls towards the center of rotation. If the mass is pulled away from the equilibrium position, then the force trying to pull it back towards equilibrium is

$$F = -mg\sin\theta, \tag{8.14}$$

where θ gives the angular displacement of the pendulum. We know that the linear displacement is $s = L\theta$ where L is the length of the pendulum (radius of the circle on which the pendulum moves). So the force pulling along the path towards equilibrium is

$$F = -mg\sin\left(\frac{s}{L}\right). \tag{8.15}$$

This does not look like Hooke's law, so in general the motion of a pendulum is not simple harmonic. At small angles, however, the sine of an angle is approximately the same as the angle itself (as long as it's measured in radians). So for **small angles**, we can write

$$F = -mg\frac{s}{L} = -\left(\frac{mg}{L}\right)s. \tag{8.16}$$

Now the equation looks like Hooke's law. The force is propotional to the linear displacement with the "spring constant" given by $k = mg/L$. Remember, however, that **this is only valid for small angles**.

Recall that the angular frequency for an object undergoing simple harmonic motion is

$$\omega = \sqrt{\frac{k}{m}}.$$

We have an expression for k for a pendulum, so we can substitute,

$$\omega = \sqrt{\frac{mg/L}{m}} = \sqrt{\frac{g}{L}}. \tag{8.17}$$

From that we can find the frequency and the period

$$f = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \tag{8.18}$$

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}}. \tag{8.19}$$

Note that the period depends only on the length of the pendulum and not on its mass or the amplitude of the motion.

Example 13.7: Measuring g

Using a small pendulum of length 0.171 m, a geophysicist counts 72.0 complete swings in a time of 6.00 s. What is the value of g in this location?

Solution: We first need to determine the period of oscillation,

$$\begin{aligned} T &= \frac{\text{time}}{\#\text{ of oscillations}} \\ T &= \frac{6.00\text{ s}}{72.0} \\ T &= 0.833\text{ s}. \end{aligned}$$

We can use this to find g,

$$\begin{aligned} T &= 2\pi\sqrt{\frac{L}{g}} \\ g &= 4\pi^2\frac{L}{T^2} \\ g &= 4\pi^2\frac{0.171\text{ m}}{(0.833\text{ s})^2} \\ g &= 9.73\text{ m/s}^2. \end{aligned}$$

8.4 Damped oscillations

So far we have assumed that the objects will continue oscillating forever. In the real world energy losses due to friction will cause the oscillating object to slow down. In this case the motion is said to be *damped.* If we consider the mass on the spring, we know that the mass will oscillate for some time but that the amplitude will decrease over time. This scenario is an *underdamped* oscillation. Suppose now we put the mass on a spring into a liquid. If the liquid is thick enough it will prevent the oscillations and simply allow the spring to come back to its equilibrium position. If the object returns to equilibrium rapidly without oscillating, then the motion is *critically damped.* If the object returns to equilibrium slowly, the motion is *overdamped.*

8.5 Waves

A wave is typically thought of as a disturbance moving through a medium. When a wave passes through a medium, the individual components of that medium oscillate about some equilibrium point, but they **do not move with the wave**. Imagine a leaf floating in a pond. You throw a pebble into the pond near the leaf. This creates a wave in the water. When the wave reaches the leaf, it causes the leaf to bo up and down, but it does not carry the leaf with it. The leaf was temporarily disturbed, but once the wave passes it goes back to its original state.

8.5.1 Types of waves

Suppose you fix one end of a string to a wall and you hold the other end. If you quickly move your hand up and down, you will create a wave (in this case a *pulse*) that travels down the string. This is a *traveling wave.* In the case of the pulse on the string, the individual bits of string move up and down as the pulse goes through. They do not move in the direction of pulse. When the disturbed medium moves perpendicular to the direction of the wave, the wave is called a *transverse wave.* A *longitudinal wave* occurs when the disturbed medium oscillates along the direction of travel of the wave. A good example of this is to alternately stretch and compress a spring. The stretched and compressed regions will move down the spring with each coil oscillating along the direction of the wave. While the coils oscillate along this direction they still do not actually move with the wave. It turns out that each point in the medium undergoes simple harmonic motion as the wave passes through.

8.5.2 Velocity of a wave

The *frequency* of a wave is determined by the frequency of the individual oscillating points. The *amplitude* of the wave is the amplitude of the oscillations. The *wavelength* of a wave is the distance between two successive points that behave identically (peak to peak, for example). The wavelength is denoted by λ. From these quantities we can determine the speed of the wave. The *wave speed* is the speed at which a particular part of the wave (like the peak) travels through the medium. Remember that speed is the displacement over time,

$$v = \frac{\Delta x}{\Delta t}.$$

We know that the wave moves a distance of one wavelength in the time it takes for one point on the wave to move through a single cycle (it's period),

$$v = \frac{\lambda}{T} = f\lambda. \tag{8.20}$$

8.5.3 Interference of waves

One interesting aspect of waves is how they interact with other waves. Two travelling waves will pass right through each other when they meet. When you throw two pebbles into the water near each other they will each create waves rippling from the point of entry. When those two waves meet they don't destroy each other. Each wave comes out of the interaction undisturbed.

At the point(s) where the two waves meet, they interact with each other in a process called *interference*. At these points, the motion of the points in space is determined by the *principle of superposition*:

> When two or more travelling waves encounter each other while moving through a medium, the resultant wave is found by adding together the displacements of the individual waves point by point.

If the peaks and troughs of two waves occur at the same place at the same time, the waves are *in phase* and the resulting interference is *constructive interference*. If the peak of one wave occurs at the same time and place as the trough of another wave then the waves are *inverted* and the resulting interference is *destructive interference*. In this case the waves completely cancel each other in the region where they interact (they will re-appear once they pass through each other).

8.5.4 Reflection of waves

Waves cannot travel in a particular medium indefinitely. Eventually the waves will reach a boundary. When the waves reach the boundary, some of the wave will be reflected and some of the wave will be transmitted. The reflected wave can sometimes be inverted with respect to the incoming wave. Consider a wave on a string that approaches a wall where the string is fixed. The string will pull on the wall as the wave hits. The wall will exert an equal and opposite force on the string (Newton's third law), pulling the string in the opposite direction. This causes the reflected wave to be inverted. If the end of the string is free to move, the reflected wave will have the same orientation as the original wave.

8.6 Sound waves

Sound waves are longitudinal waves that are caused by vibrating objects. When an object vibrates, it pushes the air near it causing alternating compression and stretching of the spacing between molecules (density) in the air. This vibration is picked up by our ears and is interpreted by our brains as sound. In a sound wave, the air molecules oscillate along the direction of travel of the wave (think of the longitudinal wave on a spring).

Sound waves can have a range of frequencies. The audible waves have frequencies between 20 and 20000 Hz. *Infrasonic* waves have frequencies below the audible range while *ultrasonic* waves have frequencies above the audible range.

8.6.1 Energy and intensity of sound waves

Sound waves are created because a vibrating object pushes air molecules. The vibrating object exerts a force on the air and so is doing work on the air. The sound wave carries that energy away from the vibrating object. For waves, we don't typically measure the total energy in the wave, but instead measure the flow of energy. The average *intensity* I of a wave on a given surface is defined as the rate at which energy flows through the surface, divided by the surface area,

$$I = \frac{1}{A}\Delta E \Delta t, \tag{8.21}$$

where the direction of energy flow is perpendicular to the surface. The rate of energy transfer is power, so we can also write this as

$$I = \frac{P}{A}. \tag{8.22}$$

The units of intensity are $\mathrm{W/m^2}$.

The faintest sounds a human ear can hear have an intensity of about $1 \times 10^{-12}\ \mathrm{W/m^2}$. This is the threshold of hearing. The loudest sounds the ear can tolerate have an intensity of about $1\ \mathrm{W/m^2}$. This is the threshold of pain. You'll notice that the intensities that a human ear can detect vary over a very wide range. The quietest sounds don't seem to us to be $1{\times}10^{12}$ times quieter than the loudest sounds because our brains us an approximately logarithmic scale to determine loudness. This is measured by the *intensity level* defined by

$$\beta = 10 \log\left(\frac{I}{I_0}\right), \tag{8.23}$$

where the constant $I_0 = 1 \times 10^{-12}\ \mathrm{W/m^2}$ is the reference intensity. β is measured in decibels (dB).

Example 14.2: Noisy grinding machine

A noisy grinding machine in a factory produces a sound intensity of 1.0×10^{-5} W/m^2. Calculate (a) the decibel level of this machine and (b) the new intensity level when a second, identical machine is added to the factory. (c) A certain number of additional machines are put into operation alongside these two. The resulting decibel level is 77.0 dB. Find the sound intensity.

Solution: (a) For a single grinder

$$\begin{aligned}
\beta &= 10\log\left(\frac{I}{I_0}\right) \\
\beta &= 10\log\left(\frac{1.0\times10^{-5}\ \mathrm{W/m^2}}{1.0\times10^{-12}\ \mathrm{W/m^2}}\right) \\
\beta &= 70.0\ \mathrm{dB}.
\end{aligned}$$

(b) With a second grinder the total intensity is 2.0×10^{-5} W/m^2. The decibel level is

$$\begin{aligned}
\beta &= 10\log\left(\frac{I}{I_0}\right) \\
\beta &= 10\log\left(\frac{2.0\times10^{-5}\ \mathrm{W/m^2}}{1.0\times10^{-12}\ \mathrm{W/m^2}}\right) \\
\beta &= 73.0\ \mathrm{dB}.
\end{aligned}$$

(c) In this case we are given the decibel level and want the intensity

$$\begin{aligned}
\beta &= 10\log\left(\frac{I}{I_0}\right) \\
77\ \mathrm{dB} &= 10\log\left(\frac{I}{1.0\times10^{-12}\ \mathrm{W/m^2}}\right) \\
10^{7.7} &= \frac{I}{1.0\times10^{-12}\ \mathrm{W/m^2}} \\
I &= 5.01\times10^{-5}\ \mathrm{W/m^2}.
\end{aligned}$$

There are five machines in all.

Many sound waves can be thought of as coming from a *point source*. A point source is small compared to the waves and emits waves symmetrically. The waves emitted by a point source are *spherical waves*; they spread in a uniform sphere. Suppose that the average power emitted by the source is P_{av}. Then the intensity at a distance r is

$$I = \frac{P_{av}}{A} = \frac{P_{av}}{4\pi r^2}. \tag{8.24}$$

The average power always remains the same, no matter the distance so we can write

$$\begin{aligned}
I_1 &= \frac{P_{av}}{4\pi r_1^2} \\
I_2 &= \frac{P_{av}}{4\pi r_2^2}.
\end{aligned}$$

Since the average power is the same, we get

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}. \tag{8.25}$$

8.6.2 The doppler effect

When a moving object is making a sound, the frequency of the sound changes as as the object moves towards or away from the observer. Think of a train blowing its whistle — the whistle changes in pitch as the train approaches and as it moves away. This is known as the *Doppler effect*.

Suppose that a source is moving through the air with velocity v_s towards a stationary observer. Since the source is moving towards the observer, in the same direction as the sound wave, the waves emitted by the source get "squished". The wavelength measured by the observer is shorter than the actual wavelength emitted by the source. During a single vibration, which lasts a time T (the period), the source moves a distance $v_sT = v_s/f_s$. The wavelength detected by the observer is shortened by this amount,

$$\lambda_o = \lambda_s - \frac{v_s}{f_s}. \tag{8.26}$$

The frequency heard by the observer is

$$f_o = \frac{v}{\lambda_o} = \frac{v}{\lambda_s - \frac{v_s}{f_s}} = \frac{v}{\frac{v}{f_s} - \frac{v_s}{f_s}}.$$

Rearranging, we get

$$f_o = f_s\left(\frac{v}{v - v_s}\right). \tag{8.27}$$

The observed frequency increases when the source move towards the observer and decreases when it moves away (v_s becomes negative).

We can do a similar analysis for the case when the source is stationary and the observer is moving. In fact, both source and observer can be moving and this is covered by the general equation

$$f_o = f_s\left(\frac{v + v_o}{v - v_s}\right). \tag{8.28}$$

The sign convention is that velocities are positive when source and observer move towards each other and negative when source and observer move away from each other.

Example 14.4: Train whistle

A train moving at a speed of 40.0 m/s sounds its whistle, which has a frequency of 5.00×10^2 Hz. Determine the frequency heard by a stationary observer as the train approaches the observer.

Solution: The velocity of sound is 345 m/s. We use the equation for the doppler effect, keeping in mind that the train is approaching the observer,

$$\begin{aligned} f_o &= f_s\left(\frac{v + v_o}{v - v_s}\right) \\ f_o &= (500\text{ Hz})\left(\frac{331\text{ m/s}}{331\text{ m/s} - 40\text{ m/s}}\right) \\ f_o &= 566\text{ Hz.} \end{aligned}$$

8.7 Standing waves

Suppose we connect one end of a string to a stationary clamp and the other end to a vibrating object. The vibrating object will move down to the end of the string and will be reflected. The reflected wave will interact with the wave originating from the object and the two waves will combine according to the principle of superposition. If the string vibrates at exactly the right frequency the wave appears to stand still so it is called a *standing wave*. A *node* occurs when the two travelling waves have the same amplitude but opposite displacement, so the net displacement is zero. Halfway between two nodes there will be an *antinode* where the string vibrates with the largest amplitude. Note that the distance between two nodes is *half* the wavelength $d_{NN} = \lambda/2$.

Suppose we fix both ends of the string, then both ends must be nodes. We can then pluck the string so that we get a single antinode over the length of the string. This is the fundamental or first harmonic and we have half a wavelength on the string. Alternatively, we could set up our wave so that there is another node in the middle of the string. This is the second harmonic and we now have a full wavelength on the string. In fact there are many node/antinode patterns we can set up on the string always keeping in mind that the ends must be nodes. In general, we can set up waves whose wavelengths satisfy the condition

$$\lambda_n = \frac{2L}{n}, \tag{8.29}$$

where n is the harmonic of the wave and can be any positive integer. The frequency of the harmonic is

$$f_n = \frac{v}{\lambda_n} = \frac{vn}{2L}. \tag{8.30}$$

The velocity of a wave on a string depends on the tension in the string F and on the mass density of the string μ,

$$v = \sqrt{\frac{F}{\mu}}. \tag{8.31}$$

This allows us to write

$$f_n = \frac{n}{2L}\sqrt{\frac{F}{\mu}}. \tag{8.32}$$

This series of frequencies forms a harmonic series.

Example 14.8: Harmonics of a stretched wire

(a) Find the frequencies of the fundamental and second harmonics of a steel wire 1.00 m long with a mass per unit length of 2.00×10^{-3} kg/m and under a tension of 80.0 N. (b) Find the wavelengths of the sound waves created by the vibrating wire. Assume the speed of sound is 345 m/s.

Solution: (a) We use the formula for frequency with $n = 1$ and $n = 2$ for the fundamental and second harmonics,

$$\begin{aligned} f_1 &= \frac{1}{2L}\sqrt{\frac{F}{\mu}} = \frac{1}{2(1.00\text{ m})}\sqrt{\frac{80.0\text{ N}}{2.00 \times 10^{-3}\text{ kg/m}}} = 100\text{ Hz} \\ f_1 &= \frac{2}{2L}\sqrt{\frac{F}{\mu}} = \frac{1}{(1.00\text{ m})}\sqrt{\frac{80.0\text{ N}}{2.00 \times 10^{-3}\text{ kg/m}}} = 200\text{ Hz}. \end{aligned}$$

(b) The frequency of the vibrating string will be transferred to the air. The wave will then move at the speed of sound,

$$\begin{aligned} \lambda_1 &= \frac{v}{f_1} = \frac{345\text{ m/s}}{100\text{ Hz}} = 3.45\text{ m} \\ \lambda_2 &= \frac{v}{f_2} = \frac{345\text{ m/s}}{200\text{ Hz}} = 1.73\text{ m} \end{aligned}$$

Standing waves can also be set up with sound waves in a pipe. Even if the end of the pipe is open, some of the sound wave will be reflected back into the pipe by the edges of the pipe. The reflected wave will interfere with the original wave and, if the frequency is right, a standing wave can be established. For pipes, the possible standing wave frequencies will depend on whether one end of the pipe is closed or if both ends are open.

If both ends are open, then there must be antinodes at either end of the pipe. The first harmonic will have a single node in middle. If the length of the pipe is L, then the wavelength is $\lambda_1 = 2L$. The second harmonic will have two nodes and a wavelength of $\lambda_2 = L$. In general, the wavelength will be $\lambda_n = 2L/n$. The frequency will be

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L}, \tag{8.33}$$

where v is the speed of sound.

For a pipe that is closed at one end, there must be a node at the closed end and an antinode at the open end. In this case, the first harmonic has only a quarter wavelength inside the pipe, $\lambda_1 = 4L$. The next possible harmonic will have one node inside the pipe, but not at the center. In this case, the wavelength is $\lambda_3 = 4L/3$. There are no even harmonics in a pipe with one end closed; only the odd harmonics are possible. In general, the wavelength is $\lambda_n = 4L/n$ where n is odd integers only. The frequency is

$$f_n = \frac{nv}{4L}. \tag{8.34}$$

8.8 Beats

Standing waves are created by interference between two waves of the same frequency, More often, waves of different frequencies will interfere with each other. When waves of different frequencies interfere, then they will alternately go in and out of phase causing periods of constructive interference followed by periods of destructive interference. If you are listening to two sound waves of different frequencies, you will hear a sound that alternates between loud and soft. This loud/soft pattern is known as *beats* and is also wave. The frequency of the beats is determined by the frquency difference of the two waves

$$f_b = |f_2 - f_1|. \tag{8.35}$$

Example: Out of tune pipes

Two pipes of equal length are each open at one end. Each has a fundamental frequency of 480 Hz when the speed of sound is 347 m/s. In one pipe the air temperature is increased so that the speed of sound is now 350 m/s. If the two pipes are sounded together, what beat frequency results?

Solution: We will need to find the new fundamental frequency of the second pipe. In order to do this, we need to know the length of the pipe. We know the frequency of the unheated pipe, so we can find the length,

$$\begin{aligned} f_1 &= \frac{v}{2L} \\ L &= \frac{v}{2f} \\ L &= \frac{347 \text{ m/s}}{2(480 \text{ Hz}} \\ L &= 0.36 \text{ m.} \end{aligned}$$

Now we can find the new fundamental frequency of the second pipe,

$$\begin{aligned} f_1 &= \frac{v}{2L} \\ f_1 &= \frac{350 \text{ m/s}}{2(0.36 \text{ m}} \\ f_1 &= 486 \text{ Hz.} \end{aligned}$$

The frequency of beats is the difference between those two frequencies,

$$\begin{aligned} f_b &= |f_1 - f_2| \\ f_b &= |486 \text{ Hz} - 480 \text{ Hz}| \\ f_b &= 6 \text{ Hz.} \end{aligned}$$

Chapter 9

Solids and Fluids

9.1 States of matter

The matter you interact with every day is typically classified as being in one of three states: solid, liquid or gas. There is also a fourth state of matter that you will not ordinarily encounter, plasma. Matter consists of molecules, which are groups of atoms. The properties of particular states of matter are determined by how molecules of a substance interact.

At low temperatures, most substances are solid. Macroscopically, solids have a definite volume and shape. In a solid, the molecules are held in (relatively) fixed positions relative to each other. There are usually electrical bonds between molecules that make it difficult for molecules to move away from each other.

As temperature increases, substances change from solid to liquid. A liquid has a definite volume, but no fixed shape. In a liquid, the molecules are weakly bound to each other and so can move within the substance with some freedom.

Further increases in temperature completely break the bonds between molecules, changing the liquid into a gas. A gas has no definite shape and no definite volume. The molecules of a gas are far from each other (relative to the size of the molecules) and very rarely interact. This means that the gas can expand to fill an volume.

At extremely high temperatures (such as those encountered inside stars) the molecules and atoms of the substance are torn apart. Positive and negatively charged particles are free to move around within the substance creating long-range electrical and magnetic forces. This is a plasma.

9.1.1 Characterizing matter

Even though substances may be in the same state and will have some broad general characteristics in common, they are by no means identical. For example, two equal masses of different substances may not take up the same volume. This property is the *density* of a substance and is defined as the mass divided by its volume,

$$\rho = \frac{M}{V}. \tag{9.1}$$

The SI unit for density is kg/m^3. The density of a liquid or solid varies slightly with temperature and pressure. The density of a gas is very sensitive to temperature and pressure. Liquids are generally, but not always, less dense than solids. Gasses are about 1000 times less dense than liquids or solids.

It is sometimes convenient to standardize density by comparing it to some standard. The *specific gravity* of a substance is the ratio of its density to the density of water at 4 °C, which is $1.0 \times 10^3\ \text{kg/m}^3$.

9.2 Deformation of solids

While a solid tends to have a definite shape, the shape can often be altered with the application of a force. While a strong enough force will permanently alter the shape, often when the force is removed, the substance will return to its original shape. This is called *elastic behaviour*.

Different substances have different elastic properties, so we will need some way to quantify or characterize this. The *stress* on a material is the force per unit area that is causing some deformation. The *strain* is a measure of the amount of deformation in the material. For small stresses, stress is proportional to the strain with the constant of proportionality depending on the properties of the material. The proportionality constant is called the *elastic modulus*.

The equation for the elasticity of a substance is similar to Hooke's law,

$$F = -k\Delta x$$

and the elastic modulus can be thought of as a spring constant. It is a measure of the stiffness of a material. A substance with a large elastic modulus is hard to deform.

9.2.1 Young's modulus

Suppose we have a long bar of cross-sectional A and length L_0 that is clamped at one end. When we apply an external force F along the bar, we can change the length of the bar. At this new length, the external force is balanced by internal forces that resist the stretch. The bar is said to be *stressed*. The *tensile stress* is the magnitude of the external force divided by the cross-sectional area. The *tensile strain* is the ratio of the change in length to the original length. Since we know that stress and strain are proportional, we have

$$\frac{F}{A} = Y\frac{\Delta L}{L_0}. \tag{9.2}$$

In this equation the proportionality constant is called *Young's modulus*. We can re-write this equation as

$$F = \left(\frac{YA}{L_0}\right)\Delta L,$$

so that it looks like Hooke's law with a spring constant of $k = YA/L_0$.

The Young's modulus depends on whether the material is being stretched or compressed. Many materials are easier to stretch than compress. The elastic response is also not quite linear and substances have an *elastic limit*. The elastic limit is the point at which the stress is no longer proportional to the strain. If stretched (compressed) beyond this limit, substances will not return to their original shape once the force is released. The *ultimate strength* is the largest stress that the substance can endure and any force beyond that reaches the substance's *breaking point*.

9.2.2 Shear modulus

Suppose we have a rectangular block of some substance. One side of the rectangle is held in a fixed position. A force is applied to the other side, parallel to the side (think of sliding the cover of a book that is sitting on a table). This is called *shear stress*. The shear stress is the ratio of the magnitude of the applied force to the area of the face being sheared. The *shear strain* is the ratio of the horizontal distance moved to the height of the object.

$$\frac{F}{A} = S\frac{\Delta x}{h}, \tag{9.3}$$

where S is the *shear modulus* of the substance. In this case, the "spring constant" is $k = SA/h$. A substance with a large shear modulus is difficult to bend.

9.2.3 Bulk modulus

Suppose we have a block of some substance and we squeeze it uniformaly with a perpendicular force from all sides. This type of squeezing is common when a substance is immersed in a fluid. The *volume stress* is the ratio of the change in the magnitude of the applied force to the surface area. The *volume strain* is the ratio of the change in volume to the original volume.

$$\frac{\Delta F}{A} = -B\frac{\Delta V}{V}, \tag{9.4}$$

where B is the *bulk modulus*. The negative sign appears so that B is positive. An increase in the external force (more squeezing) results in a decrease of the volume. Materials with a large bulk modulus are difficult to compress.

Example: Shear stress on the spine

Between each pair of vertebrae of the spine is a disc of cartilage of thickness 0.5 cm. Assume the disc has a radius of 0.04 m. The shear modulus of cartilage is 1×10^7 N/m^2. A shear force of 10 N is applied to one end of the disc while the other end is held fixed. (a) What is the resulting shear strain? (b) How far has one end of the disc moved with respect to the other end?

Solution: (a) The shear strain is caused by the shear force,

$$\begin{aligned} \text{strain} &= \frac{F}{AS} \\ \text{strain} &= \frac{10 \text{ N}}{\pi(0.04 \text{ m})^2(1 \times 10^7 \text{ N/m}^2)} \\ \text{strain} &= 1.99 \times 10^{-4}. \end{aligned}$$

(b) A shear strain is defined as the displacement over the height,

$$\begin{aligned} \text{strain} &= \frac{\Delta x}{h} \\ \Delta x &= h \times \text{strain} \\ \Delta x &= (0.5 \text{ cm})(1.99 \times 10^{-4}) \\ \Delta x &= 0.99 \ \mu\text{m}. \end{aligned}$$

9.3 Pressure and fluids

While a force can deform or break a solid, forces applied to a fluid have a different result. When a fluid is at rest, all parts of the fluid are in static equilibrium. This means that the forces are balanced for every point in the fluid. If there was some kind of a force imbalance at one point, then that part of the fluid would move.

When discussing fluids, we often don't consider a force directly, but rather use the *pressure* which is the force per unit area, since fluids (and the forces that act on them) tend to be extended over some region of space. Mathematically, the pressure is given by the formula,

$$P = \frac{F}{A}. \tag{9.5}$$

The units of pressure are newton per meter^2 or pascal (Pa).

Suppose we have some fluid sitting in equilibrium in a large container. Consider the forces acting on a piece of the fluid extending from y_1 to y_2 ($y = 0$ is the top of the fluid) and having a cross-sectional area A. There are three forces acting on this piece of fluid: the force of gravity (Mg), the force caused by the pressure of the fluid above this piece pushing down (P_1A), and the force caused by the pressure from the fluid below pushing up (P_2A). This piece of fluid is not moving, so the forces must balance,

$$P_2A - P_1A - Mg = 0. \tag{9.6}$$

We can find the mass of the water from the density $M = \rho V = \rho A(y_1 - y_2)$. Substituting into our equation, we get

$$P_2 = P_1 + \rho g(y_1 - y_2). \tag{9.7}$$

You'll notice that the force of the fluid pushing upward is larger than the force of the fluid pushing down (the difference being the weight of the fluid we're considering). For a liquid near the surface of the earth exposed to the earth's atmosphere, this equation can give us the pressure at any depth h,

$$P = P_0 + \rho g h, \tag{9.8}$$

where $P_0 = 1.013 \times 10^5$ Pa is the atmospheric pressure at sea level. The atmospheric pressure arises because air is also a fluid and the large column of air over the surface of any point on earth will exert a downward pressure on the earth.

This equation also suggests that if you change the pressure at the surface of a fluid, then the change is transmitted to every point in the fluid. This is known as *Pascal's principle*. The change in pressure will also be transmitted to the containers enclosing the fluid.

Example: Oil and Water

In a huge oil tanker, salt water has flooded an oil tank to a depth $h_2 = 5.00$ m. On top of the water is a layer of oil $h_1 = 8.00$ m deep. The oil has a density of 700 kg/m^3 and salt water has a density of 1025 kg/m^3. Find the pressure at the bottom of the tank.

Solution: The surface of the oil is exposed to air, so the pressure at that point will be atmospheric pressure. We can find the pressure at the bottom of the oil layer using our equation,

$$\begin{aligned} P_1 &= P_0 + \rho_{\text{oil}} g h_1 \\ P_1 &= 1.01 \times 10^5 \text{ Pa} + (700 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(8.00 \text{ m}) \\ P_1 &= 1.56 \times 10^5 \text{ Pa}. \end{aligned}$$

This is the pressure at the surface of the water layer. At the bottom of that layer, the pressure is,

$$\begin{aligned} P_2 &= P_1 + \rho_{\text{water}} g h_2 \\ P_2 &= 1.56 \times 10^5 \text{ Pa} + (1025 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(5.00 \text{ m}) \\ P_2 &= 2.06 \times 10^5 \text{ Pa}. \end{aligned}$$

9.4 Buoyant forces

The idea of buoyancy was discovered by the Greek mathematician Archimedes and is known as *Archimedes' principle*:

> Any object completely or partially submerged in a fluid is buoyed up by a force with magnitude equal to the weight of the fluid displaced by the object.

Basically, buoyancy is the pressure difference between the fluid below and above an object. We know that the pressure from fluid below is larger than pressure from the fluid above, so the object will feel lighter if we try to lift it (since the fluid is helping us to move the object upward).

Suppose we replace the little piece of fluid in our container with a piece of lead of the same volume. The pressure above and below the lead will not change — their difference is still equal to the mass of the fluid. The lead is denser than water, so the piece of lead is heavier and the downward force of gravity is now larger. Since the forces are no longer in equilibrium, the lead will sink. The buoyant force is due to pressure differences in the surrounding fluid and will not change if a new substance is introduced.

The buoyant force is given by,

$$B = \rho_{\text{fluid}} V_{\text{fluid}} g, \tag{9.9}$$

where V_{fluid} is the volume of fluid displaced by the object.

9.4.1 Fully submerged object

For a fully submerged object, the buoyant force pushes upwards while the force of gravity pulls the object downwards.

$$\begin{aligned} Mg - B &= Ma \\ \rho_{\text{fluid}} V_{\text{fluid}} g - \rho_{\text{object}} V_{\text{object}} g &= \rho_{\text{object}} V_{\text{object}} a \\ a &= (\rho_{\text{fluid}} - \rho_{\text{object}}) \frac{g}{\rho_{\text{object}}}. \end{aligned} \tag{9.10}$$

The acceleration will be positive (upwards) if the density of the fluid is larger than the density of the object. It will be negative (downwards) if the density of the object is larger than the density of the fluid.

9.4.2 Partially submerged object

In this case, the object is in equilibrium since it is floating in the fluid and not moving either up or down. This means that the forces must be in equilibrium.

$$\begin{aligned} B &= Mg \\ \rho_{\text{fluid}} V_{\text{fluid}} g &= \rho_{\text{object}} V_{\text{object}} g \\ \frac{\rho_{\text{object}}}{\rho_{\text{fluid}}} &= \frac{V_{\text{fluid}}}{V_{\text{object}}}. \end{aligned} \tag{9.11}$$

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Example 9.8: Weighing a crown

A bargain hunter purchases a "gold" crown at a flea market. After she gets home, she hangs it from a scale and finds its weight to be 7.48 N. She then weighs the crown while it is immersed in water and now the scale reads 6.86 N. Is the crown made of pure gold?

Solution: To determine whether the crown is actually made of gold, we need to find the density of the crown. For this, we will need both the volume and mass of the crown. When the crown is weighed in the air, we have,

$$T_{\text{air}} - mg = 0.$$

When the crown is in water, the buoyant force needs to be included,

$$T_{\text{water}} + B - mg = 0.$$

Given these two equations, then, we must have that

$$\begin{aligned} B &= T_{\text{air}} - T_{\text{water}} \\ B &= 7.48 \text{ N} - 6.86 \text{ N} \\ B &= 0.980 \text{ N}. \end{aligned}$$

The buoyant force is equal to the weight of the water displaced,

$$\begin{aligned} B &= \rho_{\text{water}} V_{\text{obj}} g \\ V_{\text{obj}} &= \frac{B}{\rho_{\text{water}} g} \\ V_{\text{obj}} &= \frac{0.980 \text{ N}}{(1.0 \times 10^3 \text{ kg/m}^3)(9.8 \text{ m/s}^2)} \\ V_{\text{obj}} &= 1.0 \times 10^{-4} \text{ m}^3. \end{aligned}$$

We can easily get the mass from the first equation,

$$\begin{aligned} m &= \frac{T_{\text{air}}}{g} \\ m &= \frac{7.48 \text{ N}}{9.8 \text{ m/s}^2} \\ m &= 0.800 \text{ kg}. \end{aligned}$$

The density of the crown is

$$\begin{aligned} \rho_{\text{crown}} &= \frac{m}{V_{\text{obj}}} \\ \rho_{\text{crown}} &= \frac{0.800 \text{ kg}}{1.0 \times 10^{-4} \text{ m}^3} \\ \rho_{\text{crown}} &= 8.0 \times 10^3 \text{ kg/m}^3. \end{aligned}$$

The density of gold is 19.3×10^3 kg/m^3, so this crown is definitely not solid gold.

9.5 Fluids in motion

When fluids move, there are two broad categories for the type of motion. *Laminar* or *streamline* motion occurs when every particle that passes a particular point moves along the same smooth path followed by previous particles passing that point. The path itself is called a streamline. During laminar motion, different streamlines will not cross. When the motion of the fluid becomes irregular, or *turbulent*, the streamlines disappear and neighbouring particles can end up moving in very different directions. In turbulent flow, you tend to see *eddy currents* (little whirlpools) and other non-linear patterns.

We will only study laminar motion in this course — turbulent motion is very complicated — and we will only consider the motion of an *ideal fluid*. The ideal fluid has the following properties:

- The fluid is non-viscous, which means there is no internal friction between adjacent particles. (The viscosity of a fluid is a measure of the amount of internal friction.)
- The fluid is incompressible, which means the density is constant.
- The fluid motion is steady, meaning that the velocity and pressure at each point does not change in time.
- The fluid moves without turbulence. This condition means that the particles have no rotational motion and no angular velocity — they only move in straight lines.

9.5.1 Equation of continuity

Suppose a fluid flows in a pipe whose cross-sectional area increases from left to right, going from A_1 at one end to A_2 at the other end. Suppose the fluid enters the pipe with a velocity v_1. The fluid entering the pipe moves a distance $\Delta x_1 = v_1 \Delta t$ in a time Δt. The mass of water contained in this region is $\Delta M_1 = \rho_{\text{water}} A_1 \Delta x_1 = \rho_{\text{water}} A_1 v_1 \Delta t$. We can write a similar equation for the mass flowing out of the other end of the pipe, $\Delta M_2 = \rho_{\text{water}} A_2 v_2 \Delta t$. Since mass is conserved (the fluid is incompressible), we must have the same amount of mass going in as is coming out, $\Delta M_1 = \Delta M_2$, or

$$A_1 v_1 = A_2 v_2. \tag{9.12}$$

This equation is known as the equation of continuity. It tells us that fluid will speed up or slow down as the area through which they flow changes. Fluid flows faster through a pipe of small area than through a pipe of large area. The product Av is also known as the flow rate.

Example 9.12: Garden hose

A water hose 2.5 cm in diameter is used by a gardener to fill a 30.0 L bucket. The gardener notices that it takes 1.0 min to fill the bucket. A nozzle with an opening of cross-sectional are 0.500 cm^2 is then attached to the hose. The nozzle is held so the water is projected horizontally from a point 1.0 m above the ground. Over what horizontal distance will the water be projected?

Solution: We first need to determine the flow rate for the water in the absence of the nozzle. We can figure this out from how long it takes to fill the bucket,

$$
\begin{aligned}
A_1 v_1 &= \frac{30\ \text{L}}{1.00\ \text{min}} \left(\frac{1000\ \text{cm}^3}{1\ \text{L}}\right) \left(\frac{1\ \text{m}}{100\ \text{cm}}\right)^3 \left(\frac{1\ \text{min}}{60\ \text{s}}\right) \\
A_1 v_1 &= 5.0 \times 10^{-4}\ \text{m}^3/\text{s}.
\end{aligned}
$$

The flow rate remains constant when the new nozzle is attached,

$$
\begin{aligned}
A_1 v_1 &= A_2 v_2 \\
v_2 &= \frac{A_1 v_1}{A_2} \\
v_2 &= \frac{5.0 \times 10^{-4}\ \text{m}^3/\text{s}}{0.5 \times 10^{-4}\ \text{m}^2} \\
v_2 &= 10.0\ \text{m/s}.
\end{aligned}
$$

This is the initial horizontal velocity of the water. Once the water leaves the hose, it is a projectile undergoing acceleration in the vertical direction (but not horizontally). We can find how long it takes for the water to hit the ground,

$$
\begin{aligned}
y &= v_{0y} t - \frac{1}{2} g t^2 \\
t &= \sqrt{\frac{2y}{g}} \\
t &= \sqrt{\frac{2(1.0\ \text{m})}{9.8\ \text{m/s}^2}} \\
t &= 0.452\ \text{s}.
\end{aligned}
$$

Now we can find the horizontal distance travelled,

$$
\begin{aligned}
x &= v_{0x} t \\
x &= (10\ \text{m/s})(0.452\ \text{s}) \\
x &= 4.52\ \text{m}.
\end{aligned}
$$

9.6 Bernoulli's equation

Suppose that the pipe with varying diameter is now angled upwards. Let's consider the work done on the fluid in the pipe. The fluid at the lower end is pushed by the fluid behind it. The fluid at the upper end is pushed by the fluid in front of it. The net work done on the fluid in the pipe then is

$$
\begin{aligned}
W &= F_1 \Delta x_1 - F_2 \Delta x_2 \\
W &= P_1 A_1 \Delta x_1 - P_2 A_2 \Delta x_2 \\
W &= P_1 V - P_2 V. \qquad (9.13)
\end{aligned}
$$

The work done on the fluid can do one of two things: it can change the kinetic energy of the fluid, and it can change the gravitational potential energy,

$$W = \Delta KE + \Delta PE. \tag{9.14}$$

Combining these two equations we have,

$$P_1 V - P_2 V = \Delta KE + \Delta PE. \tag{9.15}$$

Now we can put in expressions for the kinetic and potential energies,

$$P_1 - P_2 = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 + \rho g y_2 - \rho g y_1. \tag{9.16}$$

We can re-write this as

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2. \tag{9.17}$$

This is known as Bernoulli's equation. Note that this is not really a new concept, it is just the conservation of energy applied to a fluid. This equation is only true, however, for laminar flow.

One consequence of this equation is that faster moving fluids exert less pressure than slowly moving fluids. We can see this by considering the pipe with a varying diameter when it is horizontally level. In this case, Bernoulli's equation simplifies to

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2.$$

We know that the fluid moves faster in the narrow region, so the pressure in that region must be lower than in the wide region in order for the two sides of the equation to balance.

Example 9.13: Shootout

A nearsighted sheriff fires at a cattle rustler with his trust six-shooter. Fortunately for the rustler, the bullet misses him and penetrates the town water tank, causing a leak. If the top of the tank is open to the atmosphere, determine the speed at which the water leaves the hole when the water level is 0.500 m above the hole.

Solution: We can use Bernoulli's equation to find the velocity. Let's choose the first point to be the top of the tank and the second point will be the hole. The pressure at both of these points is just P_0, the standard atmospheric pressure. We assume that the water level drops very slowly, so the velocity of the fluid at the top of the tank is zero. Putting these into Bernoulli's equation,

$$\begin{aligned} P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 &= P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2 \\ \rho g y_1 &= \frac{1}{2}\rho v_2^2 + \rho g y_2 \\ v &= \sqrt{2g(y_1 - y_2)} \\ v &= \sqrt{2(9.8\text{ m/s}^2)(0.500\text{ m})} \\ v &= 3.13\text{ m/s}. \end{aligned}$$

Chapter 10

Thermal physics

There is a form of energy that we have to date neglected to consider in our description of objects. This is primarily because this form of energy is not typically involved in macroscopic motion. All objects have *thermal energy*, a type of energy which we intuitively detect as the object being hot or cold.

10.1 Temperature

Determining whether an object is hot or cold is rather inexact and we would like to find a more quantitative way of measuring thermal energy. We say that two objects are in *thermal contact* if energy (particularly thermal energy) can be exchanged between them. Two objects are in *thermal equilibrium* if they are in thermal contact but there is no net exchange of energy between them. The **zeroth law of thermodynamics (law of equilibrium)** states:

> If objects A and B are separately in thermal equilibrium with a third object C , then A and B are in thermal equilibrium with each other.

This law allows us to use a *thermometer* to compare the thermal energies of two objects. Suppose we want to compare the thermal energies of two objects A and B, we could put them into thermal contact and try to detect the direction of energy flow. Suppose, however, that we cannot put the two objects into thermal contact directly. We can then use a third object, C, to compare the thermal energies of A and B. We first put the thermometer into thermal contact with A until it reaches thermal equilibrium at which point we read the thermometer. We then put the thermometer into thermal contact with B until it reaches thermal equilibrium and read the temperature again. If the two readings are the same then A and B are also in thermal equilibrium. This property allows us to define temperature — if two objects are in thermal equilibrium, then they have the same temperature.

The thermometer used to measure thermal energy must have some physical property that changes with temperature. Most thermometers use the fact that substances (solids, gasses, liquids) expand as temperature increases. This physical change can usually be measured visually allowing us to put a number on the temperature. We must first, however, calibrate the thermometer. That is, we must agree on a measurement scale for temperature. The Celsius temperature scale is defined by measuring the freezing point of water which is set to be 0 °C. and the boiling point of water which is set to be 100 °C. The scale most commonly used in the US is the Fahrenheit scale. On this scale, the freezing point is at 32°F and the boiling point is at 212°F. We can convert between the two using the formula

$$T_F = \frac{9}{5}T_C + 32. \tag{10.1}$$

The temperature scale most often used by scientists is the Kelvin scale. One of the problems with both the Celsius and Fahrenheit scales is that the freezing point and boiling point of water depend not only on the temperature but on the pressure. Scientists removed the pressure dependence by observing that the pressure of all substances goes to zero at a temperature of −273.15 °C. This temperature is known as *absolute zero* and is defined to be 0 K. The second point used to define the Kelvin scale is the triple point of water. This is the temperature and pressure at which water, water vapour, and ice exist in equilibrium. This point occurs at 0.01 °C and 4.58 mm of mercury. This temperature is defined to be 273.16 K. This means that the unit size of both the celsius and kelvin scales are the same. We convert between the two using,

$$T_C = T_K - 273.15. \tag{10.2}$$

10.2 Thermal expansion

Most substances increase in volume as their temperature (thermal energy) increases. The thermal energy of an object is actually a measure of the average velocity of the constituent atoms. As temperature increases, atoms move faster. In solids and liquids, these atoms cannot actually leave the substance, so their vibrational motion increases leading to an increased separation between atoms. Macroscopically, we see this as an increase in volume. If the expansion is small compared to the object's original size, the expansion in one dimension is approximately linear with temperature,

$$\Delta L = \alpha L_0 \Delta T, \tag{10.3}$$

where ΔL is the change in length (*not volume*), L is the original length of the object, and α is the coefficient of linear expansion for a particular substance.

Example 10.3: Expansion of a railroad track

(a) A steel railroad track has a length of 30.0 m when the temperature is 0 °C. What is the length on a hot day when the temperature is 40.0 °C? (b) What is the stress caused by this expansion?

Solution: (a) The change in length due to the temperature change,

$$\begin{aligned} \Delta L &= \alpha L_0 \Delta T \\ \Delta L &= (11 \times 10^{-6}\ /^\circ\mathrm{C})(30.0\ \mathrm{m})(40.0\ ^\circ\mathrm{C}) \\ \Delta L &= 0.013\ \mathrm{m}. \end{aligned}$$

So the new length is 30.013 m.
(b) The railroad undergoes a linear expansion, so this is a tensile strain,

$$\begin{aligned} \frac{F}{A} &= Y\frac{\Delta L}{L} \\ \frac{F}{A} &= (2.0 \times 10^{11}\ \mathrm{Pa}\left(\frac{0.013\ \mathrm{m}}{30.0\ \mathrm{m}}\right) \\ \frac{F}{A} &= 8.7 \times 10^{7}\ \mathrm{Pa}. \end{aligned}$$

Since their is a linear expansion of objects with temperature, there must also be a change in their area and volume. Suppose we have a square of material with a length of L_0. Each dimension of the square will undergo linear expansion and the new area is

$$\begin{aligned} A &= L^2 \\ A &= (L_0 + \alpha L_0 \Delta T)(L_0 + \alpha L_0 \Delta T) \\ A &= L_0^2 + 2L_0^2 \alpha \Delta T + (\alpha L_0 \Delta T)^2. \end{aligned}$$

The last term in that equation will be very small, so we will ignore it,

$$A = A_0 + 2\alpha A_0 \Delta T. \tag{10.4}$$

We can re-write this so that it looks like the linear expansion equation,

$$\Delta A = 2\alpha A_0 \Delta T. \tag{10.5}$$

We define a new coefficient $\gamma = 2\alpha$ as the coefficient of area expansion.

We can perform the same type of derivation and show that the increase in volume of a substance is given by

$$\Delta V = \beta V_0 \Delta T, \tag{10.6}$$

where β is the coefficient of volume expansion and is given by $\beta = 3\alpha$.

10.3 Ideal gas law

The effect of temperature change on a gas is somewhat more complex than in solids and liquids. A gas will expand to fill a particular container no matter what the temperature. What will change instead as the temperature increases is the pressure. There is usually a fairly complex relationship between the pressure, volume and temperature of gasses, but for an *ideal gas*, we can derive a simple relationship.

An ideal gas is a gas that is maintained at low density or pressure. In an ideal gas, particles of the gas are so far apart that they rarely interact and so we can assume there are no forces acting on any of the particles and no collisions take place. Each particle of the gas moves randomly.

Since gases contain large numbers of particles, we usually count the number of particles in *moles* where one mole is 6.02×10^{23} gas particles. The number 6.02×10^{23} is known as Avogadro's number and is denoted by N_A. Avogadro's number was chosen so that the mass in grams of one mole of an element is numerically the same as the atomic mass units of the element. Carbon 12 has an atomic mass of 12 amu, so one mole of carbon 12 weighs 12 g.

For an ideal gas, the relationship between pressure, volume, and temperature is

$$PV = nRT, \tag{10.7}$$

where n is the number of moles of the substance, and R is the universal gas constant with a value of $R = 8.31$ J/mol$\cdot$K. The ideal gas law tells us that the pressure is linearly proportional to temperature and inversely proportional to the volume. As temperature increases, pressure increases. As volume increases, pressure decreases.

Example 10.6: Expanding gas

An ideal gas at 20.0 °C and a pressure of 1.50×10^5 Pa is in a container having a volume of 1.0 L. (a) Determine the number of moles of gas in the container. (b) The gas pushes against a piston, expanding to twice its original volume, while the pressure falls to atmospheric pressure. Find the final temperature.

Solution: (a) We need to be careful that all quantities are in SI units. We will need to convert the temperature to kelvins: $T = 20 + 273 = 293$ K. And we need to convert the volume to m^3: $V = 1.0\ \text{L} = 1.0 \times 10^{-3}\ \text{m}^3$. Now we can go ahead and plug the values into the gas law,

$$\begin{aligned} PV &= nRT \\ n &= \frac{PV}{RT} \\ n &= \frac{(1.50 \times 10^5\ \text{Pa})(1.0 \times 10^{-3}\ \text{m}^3)}{(8.31\ \text{J/mol}\cdot\text{K})(293\ \text{K})} \\ n &= 6.16 \times 10^{-2}\ \text{mol}. \end{aligned}$$

(b) We can find the new temperature from the gas law,

$$\begin{aligned} T &= \frac{PV}{nR} \\ T &= \frac{(1.01 \times 10^5\ \text{Pa})(2.0 \times 10^{-3}\ \text{m}^3)}{(6.16 \times 10^{-2}\ \text{mol})(8.31\ \text{J/mol}\cdot\text{K})} \\ T &= 395\ \text{K}. \end{aligned}$$