THE DERIVATIVE OF COMPOSITION: THE CHAIN RULE

Theorem 1. Let $X, Y$ be subsets of $\mathbb{R}$, and let $f: X \to Y$ and $g: Y \to \mathbb{R}$ be two functions. Suppose that $f$ is differentiable at a point $a \in X$, and that $g$ is differentiable at $f(a) \in Y$. Then $g \circ f$ is differentiable at $a$, and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Proof. Let us write the difference quotient of $g \circ f$ at the point $a$:

$$
\frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = \frac{g(f(x)) - g(f(a))}{x - a}.
$$

If $f(x) \neq f(a)$, we can write

$$
\frac{g(f(x)) - g(f(a))}{x - a} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a}.
$$

Notice that, if $f(x) \neq f(a)$, then

$$
\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} = \lim_{y \to f(a)} \frac{g(y) - g(f(a))}{y - f(a)} = g'(f(a)),
$$

after the change of variable $y = f(x)$.

(Why is that true? Recall that differentiability implies continuity. Therefore $f$ is continuous at $a$, and $g$ at $f(a)$. Hence, $\lim_{x \to a} f(x) = f(a)$. Then, by the theorem on the limit of compositions, $\lim_{x \to a} G(f(x)) = \lim_{y \to f(a)} G(y)$, for some continuous function $G$ – it will be defined below.)

Now, we want to take care of the possibility that $f(x) = f(a)$ arbitrarily close to $a$. (Think about the function $G$.) Define the function $G: Y \to \mathbb{R}$ by

$$
G(y) = \begin{cases} 
\frac{g(y) - g(f(a))}{y - f(a)}, & y \neq f(a) \\
g'(f(a)), & y = f(a).
\end{cases}
$$
$G$ is continuous on $Y$. Moreover, by checking the cases $f(x) \neq f(a)$ and $f(x) = f(a)$ separately, we see that

$$G(f(x)) \cdot \frac{f(x) - f(a)}{x - a} = \frac{g(f(x)) - g(f(a))}{x - a},$$

for all $x \in X$.

Taking the limit as $x$ approaches $a$ gives

$$\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = \lim_{x \to a} \left( G(f(x)) \cdot \frac{f(x) - f(a)}{x - a} \right) = g'(f(a)) \cdot f'(a).$$